

7.6 Product-to-Sum and Sum-to-Product Formulas

Product to Sum Formulas p. 485

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

Sum to Product Formulas p. 487

$$\sin \alpha + \sin \beta = 2 \sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin \alpha - \sin \beta = 2 \sin\left(\frac{\alpha - \beta}{2}\right) \cos\left(\frac{\alpha + \beta}{2}\right)$$

$$\cos \alpha + \cos \beta = 2 \cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos \alpha - \cos \beta = -2 \sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

Example 1: Express each of the following products as a sum containing only sines or cosines.

$$\begin{aligned}
 \text{a) } \sin^{\alpha}(6\theta) \sin^{\beta}(4\theta) &= \frac{1}{2} [\cos(6\theta - 4\theta) - \cos(6\theta + 4\theta)] \\
 &= \frac{1}{2} [\cos 2\theta - \cos 10\theta] \\
 \text{b) } \cos^{\alpha}(3\theta) \cos^{\beta}\theta &= \frac{1}{2} [\cos(3\theta - \theta) + \cos(3\theta + \theta)] \\
 &= \frac{1}{2} [\cos 2\theta + \cos 4\theta] \\
 \text{c) } \sin^{\alpha}(3\theta) \cos^{\beta}(5\theta) &= \frac{1}{2} [\sin(3\theta + 5\theta) + \sin(3\theta - 5\theta)] \\
 &= \frac{1}{2} [\sin 8\theta + \sin(-2\theta)] \\
 &= \frac{1}{2} [\sin 8\theta - \sin 2\theta]
 \end{aligned}$$

Example 2: Express each of the following sums or differences as a product of sines and/or cosines.

$$\begin{aligned}
 \text{a) } \sin^{\alpha}(5\theta) - \sin^{\beta}(3\theta) &= 2 \sin\left(\frac{5\theta - 3\theta}{2}\right) \cos\left(\frac{5\theta + 3\theta}{2}\right) \\
 &= 2 \sin\left(\frac{2\theta}{2}\right) \cos\left(\frac{8\theta}{2}\right) \\
 &= 2 \sin\theta \cos 4\theta \\
 \text{b) } \cos^{\alpha}(3\theta) + \cos^{\beta}(2\theta) &= 2 \cos\left(\frac{3\theta + 2\theta}{2}\right) \cos\left(\frac{3\theta - 2\theta}{2}\right) \\
 &= 2 \cos\left(\frac{5\theta}{2}\right) \cos\left(\frac{\theta}{2}\right)
 \end{aligned}$$