7.5 Double-angle and Half-angle Formulas

Double-angle formulas (proof page 476)

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\cos(2\theta) = 1 - 2\sin^2\theta$$

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$$

Important "rearrangements" of Double-angle formulas

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$\cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)}$$

Half-angle Formulas

$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2} \longrightarrow \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2} \longrightarrow \cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{1+\cos\alpha} \longrightarrow \tan\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \boxed{\frac{1-\cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1+\cos\alpha}}$$

use these

where $\pm$ root is determined by the quadrant of $\frac{\alpha}{2}$

EXAMPLE 1: If $\sin\theta = \frac{3}{5}, \underbrace{\frac{\pi}{2} < \theta < \pi}_{\text{in QII}}$, find the exact value of:

use expression w/ given info only if possible.

a) $\sin(2\theta) = 2\sin\theta\cos\theta$

$\sin\theta = \frac{3}{5} = \frac{y}{r}$

$x^2 + 3^2 = 5^2$

$x^2 + 9 = 25$

$x^2 = 16$

$x = -4$

b/c in Q II

$\cos\theta = \frac{-4}{5}$

$= 2\left(\frac{3}{5}\right)\left(\frac{-4}{5}\right)$

$= \frac{-24}{25}$

$\boxed{\sin(2\theta) = \frac{-24}{25}}$

b) $\cos(2\theta) = 1 - 2\sin^2\theta$

$= 1 - 2\left(\frac{3}{5}\right)^2$

$= 1 - 2\left(\frac{9}{25}\right)$

$= 1 - \frac{18}{25}$

$\boxed{\cos(2\theta) = \frac{7}{25}}$

EXAMPLE 2: Establishing Identities

a. Develop a formula for $\tan(2\theta)$ in terms of $\tan\theta$

b. Develop a formula for $\sin(3\theta)$ in terms of $\sin\theta$ *and* $\cos\theta$.

$\longrightarrow$ Know $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

a) $\tan(2\theta) = \tan(\theta + \theta)$

$= \dfrac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$

$\tan(2\theta) = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

b) $\sin(3\theta) = \sin(2\theta + \theta)$

$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$

$= (2\sin\theta\cos\theta)\cos\theta + (\cos^2\theta - \sin^2\theta)\sin\theta$

$= 2\sin\theta\cos^2\theta + \sin\theta\cos^2\theta - \sin^3\theta$

$= 3\sin\theta\cos^2\theta - \sin^3\theta$

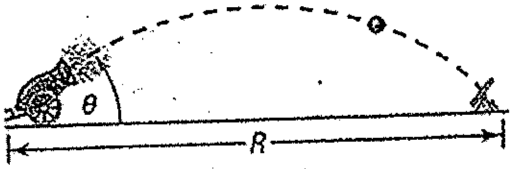
EXAMPLE 3: Write an equivalent expression for $\cos^4 \theta$ that does not involve any powers of sine or cosine greater than 1.

Not done in class
- look in book if
you care - ü

EXAMPLE 4: An object is propelled upward at an angle of θ to the horizontal with an initial velocity of v_o feet per second. See figure below. If air resistance is ignored, the range R, the horizontal distance that the object travels , is given by the function

$$R(\theta) = \frac{1}{16} v_0^2 \sin\theta \cos\theta$$

Know $\sin(2\theta) = 2\sin\theta\cos\theta$



$$\frac{1}{16} v_0^2 \sin\theta\cos\theta = \frac{1}{16} v_0^2 \left(\frac{2}{2}\right) \sin\theta\cos\theta$$

a. Show that $R(\theta) = \frac{1}{32} v_0^2 \sin(2\theta)$

$$= \frac{1}{2} \cdot \frac{1}{16} v_0^2 \, 2\sin\theta\cos\theta$$

$$= \frac{1}{32} v_0^2 \sin(2\theta) \checkmark$$

b. Find the angle θ for which R is a maximum. $R(\theta) = \frac{1}{32} v_0^2 \sin(2\theta)$

to get max out of $\sin(2\theta)$
we know $\sin(2\theta) = 1$

$2\theta = 90^\circ = \frac{\pi}{2}$

$\theta = 45^\circ = \frac{\pi}{4}$

So to get the maximum range, the angle to the horizontal must be 45°.

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}} \qquad \sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

where sign of root determined by $\frac{\alpha}{2}$

No negative arguments! b/c sine is odd

EXAMPLE 5: Use a Half-angle Formula to find the exact value of:

a. cos 15°

15° in Q I use + root

$15^\circ = \frac{30^\circ}{2} \qquad \alpha = 30^\circ$

$$\cos 15^\circ = \sqrt{\frac{1+\cos 30^\circ}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}} = \sqrt{\frac{2+\sqrt{3}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \boxed{\frac{\sqrt{2+\sqrt{3}}}{2}}$$

b. sin(−15°) = $-\sin 15^\circ$

$$= -\sqrt{\frac{1-\cos 30^\circ}{2}} = -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = -\sqrt{\frac{\frac{2-\sqrt{3}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = \frac{-\sqrt{2-\sqrt{3}}}{2}$$

EXAMPLE 6: If $\cos\alpha = \frac{-3}{5}$, $\pi < \alpha < \frac{3\pi}{2}$ (in Q III), find the exact value of:

Need location of $\frac{\alpha}{2}$

$\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4}$ $\quad \frac{\alpha}{2}$ is in Q II

$\sin\theta > 0$

$\csc\theta > 0$

a. $\sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}}$

$$= \sqrt{\frac{1- -\frac{3}{5}}{2}} = \sqrt{\frac{\frac{5}{5}+\frac{3}{5}}{2}} = \sqrt{\frac{\frac{8}{5}}{2}} = \sqrt{\frac{\cancel{8}^{4}}{5}\cdot\frac{1}{\cancel{2}}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}} = \boxed{\frac{2\sqrt{5}}{5}}$$

b. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$

$$= -\sqrt{\frac{\frac{5}{5}+\frac{-3}{5}}{2}} = -\sqrt{\frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{2}}} = -\sqrt{\frac{1}{5}} = \frac{-1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{-\sqrt{5}}{5}}$$

c. $\tan\frac{\alpha}{2}$

$$= -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = -\sqrt{\frac{1+\frac{3}{5}}{1-\frac{3}{5}}} = -\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}} = -\sqrt{\frac{\cancel{8}^{4}}{\cancel{5}}\cdot\frac{\cancel{5}}{\cancel{2}}} = -\sqrt{4} = \boxed{-2}$$