θ = theta α = alpha β = beta γ = gamma

7.4 Sum and Difference Formulas

Sum and Difference Formulas for the Cosine Function pf 466-467

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Identities derived from the above formulas: pf. 468

$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$

Sum and Difference Formulas for the Sine Function pf. 469

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Sum and Difference Formulas for the Tangent Function pf. 471

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

can only use where $\tan\theta$ is defined

- exclude odd integer multiples of $\frac{\pi}{2}$

EXAMPLE 1: Find the exact value of cos 75°

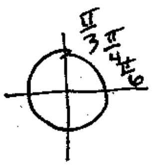
$$\cos 75^\circ = \cos(\overset{\alpha}{45^\circ} + \overset{\beta}{30^\circ})$$

$$\sqrt{2}\,(\sqrt{3}-1)$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

EXAMPLE 2: Find the exact value of $\cos\frac{\pi}{12}$ $\qquad \alpha - \beta = \frac{\pi}{12}$



$$\cos(\alpha-\beta) = \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

EXAMPLE 3: Find the exact value of $\sin\frac{7\pi}{12}$

$$\sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) =$$

EXAMPLE 4: Find the exact value of sin 80° cos 20° - cos 80° sin 20° $= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$\alpha = 80^\circ$

$\beta = 20^\circ$

$$= \sin(80^\circ - 20^\circ)$$

$$= \sin 60^\circ$$

$$= \frac{\sqrt{3}}{2}$$

QII $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ QIII

EXAMPLE 5: If it is known that $\sin\alpha = \frac{4}{5}, \frac{\pi}{2} < \alpha < \pi$, and that $\sin\beta = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}, \pi < \beta < \frac{3\pi}{2}$, find the exact value of :

a) $\cos\alpha$ $\qquad$ b) $\cos\beta$ $\qquad$ c) $\cos(\alpha+\beta)$ $\qquad$ d) $\sin(\alpha+\beta)$

QII

a) $\sin\alpha = \frac{4}{5} = \frac{y}{r}$

$x^2+y^2=r^2$

$x^2+4^2=5^2$

$x^2+16=25$

$x^2=9$

$x=-3$

b/c in QII

$\cos\alpha = \frac{x}{r} = \frac{-3}{5}$

b) $\sin\beta = \frac{-2}{\sqrt{5}} = \frac{y}{r}$

$x^2+y^2=r^2$

$x^2+(-2)^2=(\sqrt{5})^2$

$x^2+4=5$

$x^2=1$

$x=-1$

$\cos\beta = \frac{x}{r} = \frac{-1}{\sqrt{5}}$

$= \frac{-\sqrt{5}}{5}$

c) $\cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \frac{-3}{5}\cdot\frac{-\sqrt{5}}{5} - \frac{4}{5}\cdot\frac{-2\sqrt{5}}{5}$

$= \frac{3\sqrt{5}}{25} + \frac{8\sqrt{5}}{25}$

$= \frac{11\sqrt{5}}{25}$

d) $= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{4}{5}\cdot\frac{-\sqrt{5}}{5} + \frac{-3}{5}\cdot\frac{-2\sqrt{5}}{5}$

$= \frac{-4\sqrt{5}}{25} + \frac{6\sqrt{5}}{25}$

$= \frac{2\sqrt{5}}{25}$

$\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

EXAMPLE 6: Establish the identity: $\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

$\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta}{\sin\alpha\sin\beta}$

$= \frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} + \frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}$

$= \cot\alpha\cot\beta + 1$

$\therefore \frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

EXAMPLE 7: Prove the identity $\tan(\theta+\pi) = \tan\theta$

$\tan(\overset{\alpha}{\theta}+\overset{\beta}{\pi}) = \frac{\tan\theta + \tan\pi}{1-\tan\theta\tan\pi}$

$= \frac{\tan\theta + 0}{1-\tan\theta(0)}$

$= \frac{\tan\theta}{1}$

$= \tan\theta$

$\therefore \tan(\theta+\pi) = \tan\theta$

$$\tan\left(\theta+\frac{\pi}{2}\right) = -\cot\theta$$

EXAMPLE 8: Prove the identity: $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$

$$\tan\left(\theta+\frac{\pi}{2}\right) = \frac{\sin\left(\theta+\frac{\pi}{2}\right)}{\cos\left(\theta+\frac{\pi}{2}\right)}$$

$$= \frac{\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}}$$

$$= \frac{\sin\theta(0) + \cos\theta(1)}{\cos\theta(0) - \sin\theta(1)}$$

$$= \frac{\cos\theta}{-\sin\theta}$$

$$= -\cot\theta$$

$$\therefore \tan\left(\theta+\frac{\pi}{2}\right) = -\cot\theta$$

EXAMPLE 9: Find the exact value of $\sin\left(\cos^{-1}\frac{1}{2} + \sin^{-1}\frac{3}{5}\right)$

$\cos^{-1}\frac{1}{2} = \alpha$

$\cos\alpha = \frac{1}{2} = \frac{x}{r}$

in Q I

$x^2+y^2=r^2$

$1^2+y^2=2^2$

$1+y^2=4$

$y^2=3$

$y=\sqrt{3}$

$\sin\alpha = \frac{y}{r}$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\sin^{-1}\frac{3}{5} = \beta$

$\sin\beta = \frac{3}{5} = \frac{y}{r}$

in Q I

$x^2+3^2=5^2$

$x^2+9=25$

$x^2=16$

$x=4$

$\cos\beta = \frac{4}{5}$

$= \sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{4}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)$

$= \frac{4\sqrt{3}}{10} + \frac{3}{10}$

$= \frac{3+4\sqrt{3}}{10}$

EXAMPLE 10: Write $\sin(\sin^{-1} u + \cos^{-1} v)$ as an algebraic expression containing u and v (that is, without any trigonometric functions). Give the restrictions on u and v.

$$\sin(\sin^{-1}u + \cos^{-1}v) = \sin(\alpha + \beta)$$

$-1 \le u \le 1$

$\sin^{-1} u = \alpha$

$\sin\alpha = u$

$\sin^2\alpha + \cos^2\alpha = 1$

$1 - \sin^2\alpha = \cos^2\alpha$

$\pm\sqrt{1-\sin^2\alpha} = \cos\alpha$

always + on right
choose + root

$\cos^{-1} v = \beta$

$\cos\beta = v$

$-1 \le v \le 1$

$\sin^2\beta + \cos^2\beta = 1$

$\sin^2\beta = 1 - \cos^2\beta$

$\sin\beta = \pm\sqrt{1-\cos^2\beta}$

$\sin\beta$ always + in top $\frac{1}{2}$ of circle

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$= uv + \cos\alpha\sin\beta$$

$$= uv + \left(\sqrt{1-\sin^2\alpha}\right)\left(\sqrt{1-\cos^2\beta}\right)$$

$$= uv + \left(\sqrt{1-u^2}\right)\left(\sqrt{1-v^2}\right)$$