7.3 Trigonometric Identities

Definition of identity: 2 functions are said to be identically equal if $f(x) = g(x)$ for every value of x, for which both functions are defined. Such an equation is called an identity.

Examples:

$(x+1)^2 = x^2 + 2x + 1$ $\quad$ $(a-b)(a+b) = a^2 - b^2$ $\quad$ $\sin^2\theta + \cos^2\theta = 1$ $\quad$ $\csc x = \dfrac{1}{\sin x}$

Non-examples: also called – conditional equations

$\sin\theta = \cos\theta$ $\qquad$ $2x + 5 = 0$

Established identities: (KNOW THEM)

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\qquad$ $\csc\theta = \dfrac{1}{\sin\theta}$ $\qquad$ $\sin^2\theta + \cos^2\theta = 1$

$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $\qquad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\qquad$ $1 + \tan^2\theta = \sec^2\theta$

$\cot\theta = \dfrac{1}{\tan\theta}$ $\qquad$ $1 + \cot^2\theta = \csc^2\theta$

$\sin(-\theta) = -\sin\theta$ $\qquad$ $\tan(-\theta) = -\tan\theta$ $\qquad$ $\cos(-\theta) = \cos\theta$

$\csc(-\theta) = -\csc\theta$ $\qquad$ $\cot(-\theta) = -\cot\theta$ $\qquad$ $\sec(-\theta) = \sec\theta$

Some techniques used to establish identities. "tricks of the trade"

1. Multiply by a well-chosen one.
2. Write the expression over a common denominator.
3. Rewrite the expression in terms of cosine and sine only.
4. Factor the expression.
5. Keep your goal in mind!

Example 1: Using Algebraic Techniques to Simplify Trigonometric Expressions

a) Simplify $\frac{\cot\theta}{\csc\theta}$ by rewriting each trigonometric function in terms of sine and cosine functions.

$$\frac{\cot\theta}{\csc\theta} = \frac{\frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{\cos\theta}{\sin\theta} \div \frac{1}{\sin\theta} = \frac{\cos\theta}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{1} = \cos\theta$$

b) Show that $\frac{\cos\theta}{1+\sin\theta} = \frac{1-\sin\theta}{\cos\theta}$ by multiplying the numerator and denominator by $1-\sin\theta$.

$$\frac{\cos\theta}{1+\sin\theta} = \frac{\cos\theta(1-\sin\theta)}{(1+\sin\theta)(1-\sin\theta)}$$

$$= \frac{\cos\theta(1-\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cancel{\cos\theta}(1-\sin\theta)}{\cos^{\cancel{2}}\theta}$$

$$= \frac{1-\sin\theta}{\cos\theta}$$

"Aside work"

$\sin^2\theta+\cos^2\theta = 1$

$\cos^2\theta = 1-\sin^2\theta$

c) Simplify $\frac{1+\sin u}{\sin u} + \frac{\cot u - \cos u}{\cos u}$ by rewriting the expression over a common denominator.

$$\frac{1+\sin u}{\sin u} + \frac{\cot u - \cos u}{\cos u} = \frac{\cos u(1+\sin u)}{\cos u \sin u} + \frac{(\cot u - \cos u)\sin u}{\cos u \sin u}$$

$$= \frac{\cos u + \cancel{\cos u \sin u} + \cot u \sin u - \cancel{\cos u \sin u}}{\cos u \sin u}$$

$$= \frac{\cos u + \cot u \sin u}{\cos u \sin u}$$

$$= \frac{\cos u + \frac{\cos u}{\cancel{\sin u}} \cdot \cancel{\sin u}}{\cos u \sin u}$$

$$= \frac{\cos u + \cos u}{\cos u \sin u} = \frac{2\cancel{\cos u}}{\cancel{\cos u}\sin u} = \frac{2}{\sin u}$$

d) Simplify $\frac{\sin^2 v - 1}{\tan v \sin v - \tan v}$ by factoring.

$$\frac{\sin^2 v - 1}{\tan v \sin v - \tan v} = \frac{\cancel{(\sin v - 1)}(\sin v + 1)}{\tan v \cancel{(\sin v - 1)}} = \frac{\sin v + 1}{\tan v}$$

Vertical format "Math essay"

Intro - Thesis : What are establishing ?
Body - proof - one step at time
choose a side to work on
(complicated side best)
Conclusion - $\therefore$ restate intro

Example 2: Establish the identity $\csc\theta \cdot \tan\theta \stackrel{?}{=} \sec\theta$

$$\csc\theta \cdot \tan\theta = \sec\theta$$

$$\csc\theta \cdot \tan\theta = \frac{1}{\cancel{\sin\theta}} \cdot \frac{\cancel{\sin\theta}}{\cos\theta}$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

$$\therefore \csc\theta\tan\theta = \sec\theta$$

Example 3: Establish the identity ~~$\sin^2\theta + \cos^2\theta = 1$~~ $\sin^2(-\theta) + \cos^2(-\theta) = 1$

$$\sin^2(-\theta) + \cos^2(-\theta) = [\sin(-\theta)]^2 + [\cos(-\theta)]^2$$

$$= [-\sin\theta]^2 + [\cos\theta]^2$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

$$\therefore \sin^2(-\theta) + \cos^2(-\theta) = 1$$

Example 4: Establish the identity $\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$

$$\frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \frac{[\sin(-\theta)]^2 - [\cos(-\theta)]^2}{\sin(-\theta) - \cos(-\theta)}$$

$$= \frac{[-\sin\theta]^2 - [\cos\theta]^2}{-\sin\theta - \cos\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{-\sin\theta - \cos\theta}$$

$$= \frac{(\sin\theta - \cos\theta)\cancel{(\sin\theta + \cos\theta)}}{-1\cancel{(\sin\theta + \cos\theta)}}$$

$$= \frac{\sin\theta - \cos\theta}{-1}$$

$$= -\sin\theta + \cos\theta$$

$$= \cos\theta - \sin\theta$$

$$\therefore \frac{\sin^2(-\theta) - \cos^2(-\theta)}{\sin(-\theta) - \cos(-\theta)} = \cos\theta - \sin\theta$$

Example 5: Establish the identity $\frac{1+\tan u}{1+\cot u} = \tan u$

$$\frac{1+\tan u}{1+\cot u} = \tan u$$

$$\frac{1+\tan u}{1+\cot u} = \frac{1+\tan u}{1+\frac{1}{\tan u}}$$

"Aside work"

$$\frac{\tan u}{\tan u} + \frac{1}{\tan u} = \frac{\tan u+1}{\tan u}$$

$$= \frac{1+\tan u}{\frac{\tan u+1}{\tan u}}$$

$$= 1+\tan u \div \frac{\tan u+1}{\tan u}$$

$$= \cancel{1+\tan u} \cdot \frac{\tan u}{\cancel{\tan u+1}}$$

$$= \tan u$$

$$\therefore \frac{1+\tan u}{1+\cot u} = \tan u$$

Example 6: Establish the identity: $\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$

$$\frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta\sin\theta}{\sin\theta(1+\cos\theta)} + \frac{(1+\cos\theta)(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{\sin^2\theta + \cos^2\theta + 1 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1+1+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2+2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2\cancel{(1+\cos\theta)}}{\sin\theta\cancel{(1+\cos\theta)}}$$

$$= \frac{2}{\sin\theta}$$

$$= 2\csc\theta$$

$$\therefore \frac{\sin\theta}{1+\cos\theta} + \frac{1+\cos\theta}{\sin\theta} = 2\csc\theta$$

Example 7: Establish the identity: $\frac{\tan\theta+\cot\theta}{\sec\theta\csc\theta} = 1$

$$\frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = 1$$

$$\frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\cos\theta} \cdot \frac{1}{\sin\theta}}$$

$$= \frac{\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}}{\frac{1}{\cos\theta\sin\theta}}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cancel{\cos\theta\sin\theta}} \cdot \frac{\cancel{\cos\theta\sin\theta}}{1}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{1}$$

$$= \frac{1}{1}$$

$$= 1$$

$$\therefore \frac{\tan\theta + \cot\theta}{\sec\theta\csc\theta} = 1$$

θ = theta α = alpha β = beta γ = gamma

7.4 Sum and Difference Formulas

Sum and Difference Formulas for the Cosine Function

proof pg. 466-467

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Identities derived from the above formulas: proof p. 468

$$\cos\left(\tfrac{\pi}{2}-\theta\right) = \sin\theta$$

$$\sin\left(\tfrac{\pi}{2}-\theta\right) = \cos\theta$$

Sum and Difference Formulas for the Sine Function

proof p. 469

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Sum and Difference Formulas for the Tangent Function

proof pg. 471

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

can only use where $\tan\theta$ is defined
so not defined on odd integer multiples of $\frac{\pi}{2}$

"no calc"

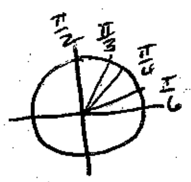
EXAMPLE 1: Find the exact value of $\cos 75^\circ$

$$\cos(75^\circ) = \cos(\overset{\alpha}{45^\circ} + \overset{\beta}{30^\circ})$$

$$= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

EXAMPLE 2: Find the exact value of $\cos\frac{\pi}{12}$



$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{4\pi}{12} - \frac{3\pi}{12}\right)$$

$$= \cos\left(\overset{\alpha}{\frac{\pi}{3}} - \overset{\beta}{\frac{\pi}{4}}\right)$$

$$= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

EXAMPLE 3: Find the exact value of $\sin\frac{7\pi}{12}$

$$\sin\frac{7\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$$

EXAMPLE 4: Find the exact value of $\sin \overset{\alpha}{80^\circ} \cos \overset{\beta}{20^\circ} - \cos \overset{\alpha}{80^\circ} \sin \overset{\beta}{20^\circ}$

$$= \sin(80^\circ - 20^\circ)$$

$$= \sin 60^\circ = \frac{\sqrt{3}}{2}$$

QII QIII

EXAMPLE 5: If it is known that $\sin\alpha = \frac{4}{5}, \frac{\pi}{2} < \alpha < \pi$, and that $\sin\beta = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}, \pi < \beta < \frac{3\pi}{2}$, find the exact value of :

a) $\cos\alpha$

$\cos\alpha = \frac{x}{r}$

$\sin\alpha = \frac{y}{r} = \frac{4}{5}$

$y = 4 \quad r = 5$

$x^2 + y^2 = r^2$

$x^2 + 4^2 = 5^2$

$x^2 + 16 = 25$

$x^2 = 9$

$x = -3$

b/c in QII

$\boxed{\cos\alpha = \frac{-3}{5}}$

b) $\cos\beta$

$\cos\beta = \frac{x}{r}$

$\sin\alpha = \frac{y}{r} = \frac{-2}{\sqrt{5}}$

$y = -2 \quad r = \sqrt{5}$

$x^2 + (-2)^2 = (\sqrt{5})^2$

$x^2 + 4 = 5$

$x^2 = 1$

$x = -1$

b/c in QIII

$\cos\beta = \frac{-1}{\sqrt{5}}$

$\boxed{\cos\beta = \frac{-\sqrt{5}}{5}}$

c) $\cos(\alpha + \beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\frac{-3}{5}\left(\frac{-\sqrt{5}}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{-2\sqrt{5}}{5}\right)$

$= \frac{3\sqrt{5}}{25} + \frac{8\sqrt{5}}{25}$

$= \boxed{\frac{11\sqrt{5}}{25}}$

d) $\sin(\alpha + \beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \frac{4}{5}\left(\frac{-\sqrt{5}}{5}\right) + \left(\frac{-3}{5}\right)\left(\frac{-2\sqrt{5}}{5}\right)$

$= \frac{-4\sqrt{5}}{25} + \frac{6\sqrt{5}}{25}$

$= \boxed{\frac{2\sqrt{5}}{25}}$

EXAMPLE 6: Establish the identity: $\frac{\cos(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\alpha\cot\beta + 1$

EXAMPLE 7: Prove the identity $\tan(\theta + \pi) = \tan\theta$