Pre-Calculus 7.1 The Inverse Sine, Cosine, and Tangent Functions

One-to-one review:

One-to-one : A function is 1-to-1 if for every output there is exactly one input.

If a function is 1-to 1 then it has an inverse. If not 1-to-1, restrict the domain until 1-to-1.

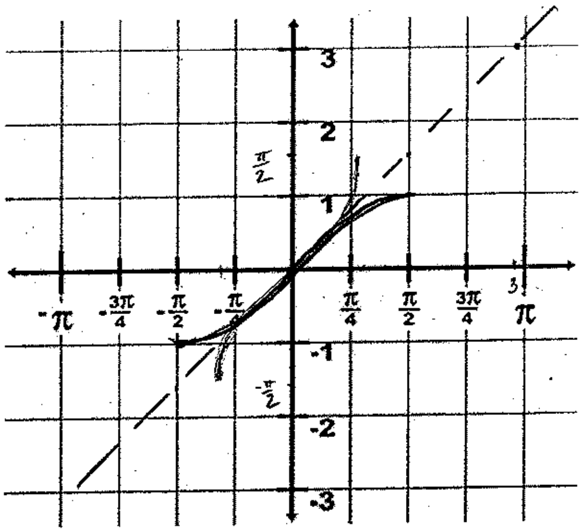
<u>Properties of inverses</u>

1. Graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y=x$.
2. Domain of $f(x)$ = Range of $f^{-1}(x)$ and Range of $f(x)$ = Domain of $f^{-1}(x)$.
3. $f \circ f^{-1} = f(f^{-1}(x)) = x$ for every x in the domain of $f^{-1}(x)$.
   $f^{-1} \circ f = f^{-1}(f(x)) = x$ for every x in the domain of $f(x)$.
4. If $y=f(x)$ has an inverse, $x=f(y)$ (switch x and y) implicit definition. Solve for y to get explicit def'n $y=f^{-1}(x)$.

Inverse Sine:

$y=\sin x$

x	y
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0



$y=\sin x$

x	y
$-\frac{\pi}{2}$	-1
0	0
$\frac{\pi}{2}$	1

D: $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

R: $-1 \le y \le 1$

$y=\sin^{-1}x$

x	y
-1	$-\frac{\pi}{2}$
0	0
1	$\frac{\pi}{2}$

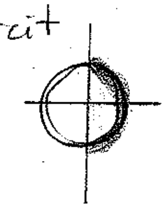
$x = \sin y$ implicit

$y=\sin^{-1}x$ explicit

$y = \arcsin x$ "

D: $-1 \le x \le 1$

R: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



y is the angle whose sine is x. Output of an inverse trig fxn is an angle.

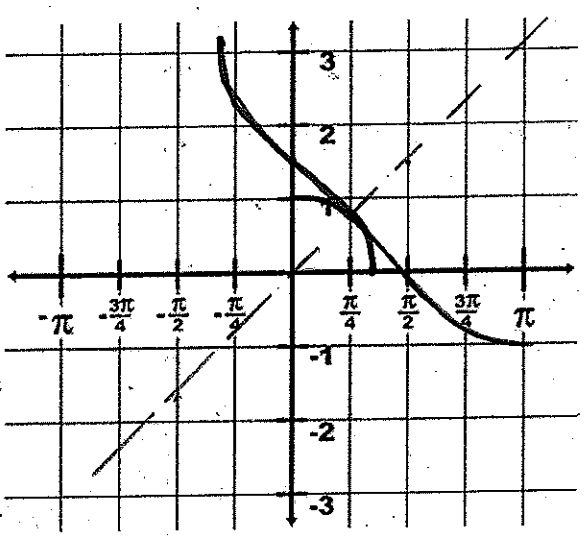
Inverse Cosine:

$y=\cos x$

x	y
0	1
$\frac{\pi}{2}$	0
π	-1

$y=\cos^{-1}x$

x	y
1	0
0	$\frac{\pi}{2}$
-1	π

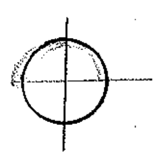


$y=\cos x$ D: $0 \le x \le \pi$

R: $-1 \le y \le 1$

$y=\cos^{-1}x = \arccos x$ D: $-1 \le x \le 1$

R: $0 \le y \le \pi$

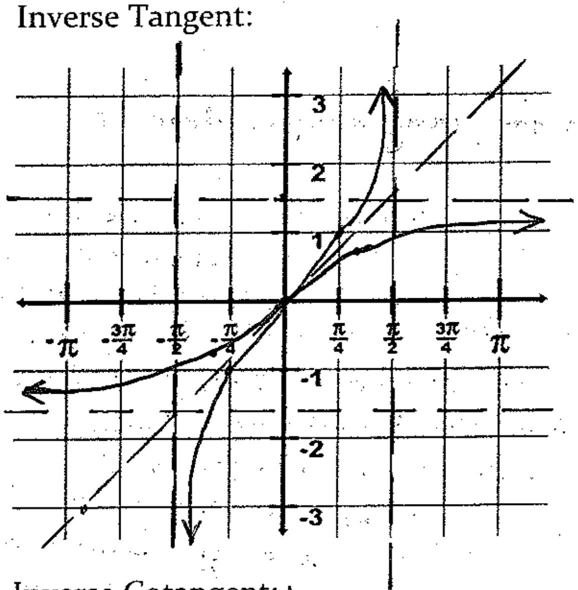


per. 1

blue ↓ y = tan x

x	y
$-\frac{\pi}{2}$	ud
$-\frac{\pi}{4}$	-1
0	0
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	ud

Inverse Tangent:

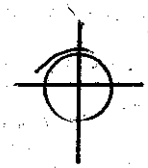


$y = \tan x$

$D: -\frac{\pi}{2} < x < \frac{\pi}{2}$ $\quad R: \mathbb{R}$

$y = \tan^{-1} x = \arctan x$

$D: \mathbb{R}$ $\quad R: -\frac{\pi}{2} < y < \frac{\pi}{2}$



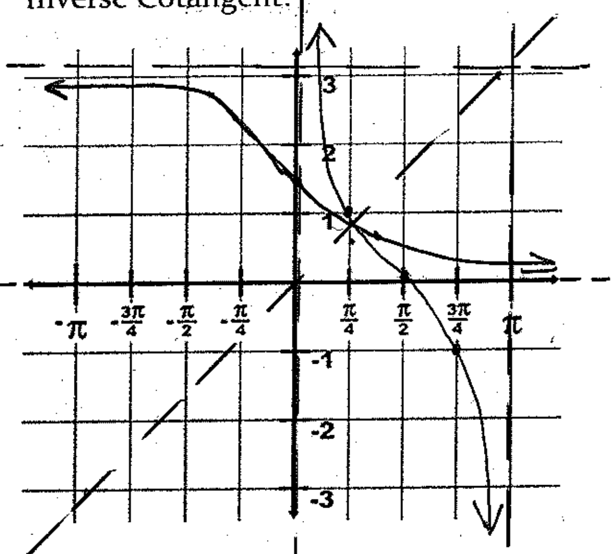
$y = \tan^{-1} x$

x	y
ud	$-\frac{\pi}{2}$
-1	$-\frac{\pi}{4}$
0	0
1	$\frac{\pi}{4}$
ud	$\frac{\pi}{2}$

y = cot x

x	y
0	ud
$\frac{\pi}{4}$	1
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	-1
π	ud

Inverse Cotangent:

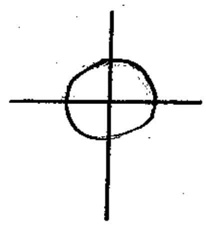


$y = \cot x$

$D: 0 < x < \pi$ $\quad R: \mathbb{R}$

$y = \cot^{-1} x = \operatorname{arccot} x$

$D: \mathbb{R}$ $\quad R: 0 < y < \pi$



$y = \cot^{-1} x$

x	y
ud	0
1	$\frac{\pi}{4}$
0	$\frac{\pi}{2}$
-1	$\frac{3\pi}{4}$
ud	π

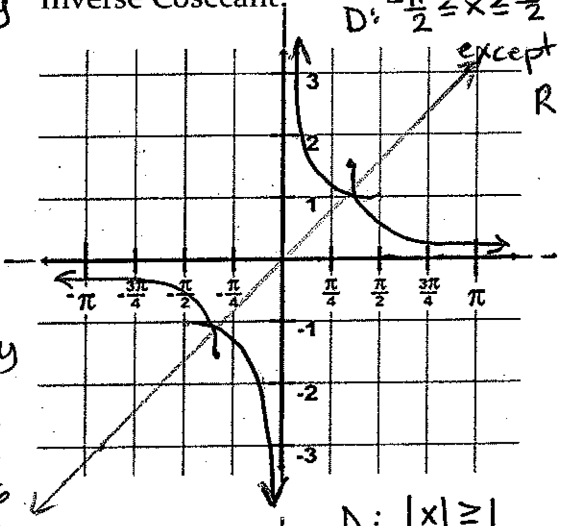
csc x = y

x	y
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{6}$	-2
0	ud
$\frac{\pi}{6}$	2
$\frac{\pi}{2}$	1

Inverse Cosecant:

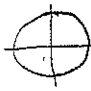
$D: -\frac{\pi}{2} \le x \le \frac{\pi}{2}$ except $x \ne 0$

$R: |y| \ge 1$



$\csc^{-1} x = y$

-1	$-\frac{\pi}{2}$
-2	$-\frac{\pi}{6}$
ud	0
2	$\frac{\pi}{6}$
1	$\frac{\pi}{2}$



$D: |x| \ge 1$

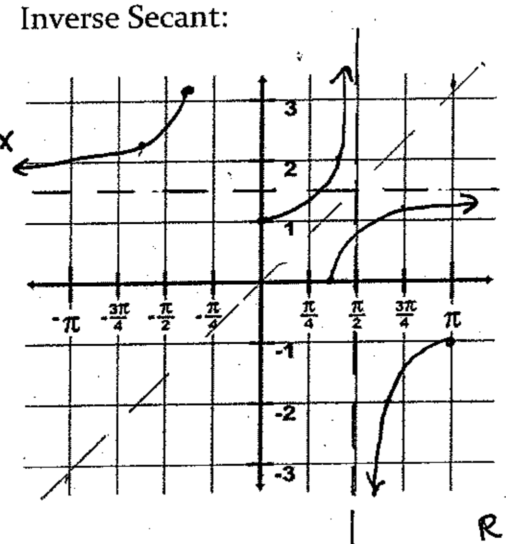
$R: -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

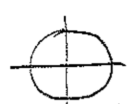
except $y \ne 0$

Inverse Secant:

$y = \sec^{-1} x$

x	y
1	0
2	$\frac{\pi}{3}$
ud	$\frac{\pi}{2}$
-2	$\frac{2\pi}{3}$
-1	π





y = sec x

x	y
0	1
$\frac{\pi}{3}$	2
$\frac{\pi}{2}$	ud
$\frac{2\pi}{3}$	-2
π	-1

D:
$0 \le x \le \pi$
except $x \ne \frac{\pi}{2}$

$R: |y| \ge 1$

Example 1: Find the "exact value" of $\sin^{-1} 1 = \Theta$ (no calc)

"you want the angle whose sine is 1"

$$\sin^{-1} 1 = \Theta$$

means $\sin\Theta = 1$

$$\sin\frac{\pi}{2} = 1 \quad \therefore \sin^{-1} 1 = \frac{\pi}{2}$$

Example 2: Find the exact value of $\sin^{-1}\left(-\frac{1}{2}\right) = \Theta$

means $\sin\Theta = \frac{-1}{2}$

$$\sin\frac{-\pi}{6} = \frac{-1}{2}$$

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

Example 3: Find the "use calc" approximate value of (2 decimal places – in radians!)

a) $\sin^{-1}\left(\frac{1}{3}\right) \approx \cancel{3}.34$

b) $\sin^{-1}\left(-\frac{1}{4}\right) \approx -.25$

D of sine $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Example 4: Find the exact value of each of the following composite functions:

a) $\sin^{-1}\left(\sin\frac{\pi}{8}\right) = \frac{\pi}{8}$

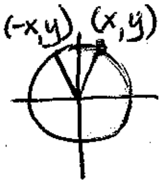
$-\frac{\pi}{2} \le \frac{\pi}{8} \le \frac{\pi}{2}$?

True - so we can do composition.

b) $\sin^{-1}\left(\sin\frac{5\pi}{8}\right) = \sin^{-1}\left(\sin\frac{3\pi}{8}\right) = \frac{3\pi}{8}$

$-\frac{\pi}{2} \le \frac{5\pi}{8} \le \frac{\pi}{2}$

$(-x,y)$, (x,y)



$\sin\frac{5\pi}{8} = \sin\frac{3\pi}{8}$

$\frac{5\pi}{8}$ is not in the domain but

$\downarrow$ in domain

Example 5: Find the exact value of each of the following composite functions:

D of $\sin^{-1}x$ $-1 \le x \le 1$

a) $\sin(\sin^{-1} 0.8) = .8$

$-1 \le .8 \le 1$?

yes

b) $\sin(\sin^{-1} 1.8)$ = undefined

$-1 \le 1.8 \le 1$?

No/False

No symmetry / No trick

Example 6: Find the exact value of $\cos^{-1} 0 = \theta$

$\cos\theta = 0$

$\downarrow$

$\cos\frac{\pi}{2} = 0$

$\downarrow$

$\cos^{-1} 0 = \frac{\pi}{2}$

Example 7: Find the exact value of $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \theta$

$\cos\theta = -\frac{\sqrt{2}}{2}$

$\downarrow$

$\cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$\downarrow$

$\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

$\cos\theta \quad D: 0 \le \theta \le \pi$ $\qquad\qquad$ $\cos^{-1}x \quad D: -1 \le x \le 1$

Example 8: Find the exact value of:

a) $\cos^{-1}\left(\cos\frac{\pi}{12}\right)$ $\quad$ b) $\cos[\cos^{-1}(-0.4)]$ $\quad$ c) $\cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right]$ $\quad$ d) $\cos(\cos^{-1}\pi)$

a) $\cos^{-1}\left(\cos\frac{\pi}{12}\right) = \frac{\pi}{12}$

$0 \le \frac{\pi}{12} \le \pi$?

True

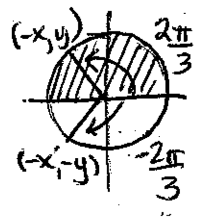
b) $\cos\left[\cos^{-1}(-.4)\right] = -.4$

$-1 \le -.4 \le 1$?

True

c) $\cos^{-1}\left[\cos\left(-\frac{2\pi}{3}\right)\right] = \cos^{-1}\left[\cos\frac{2\pi}{3}\right] = \frac{2\pi}{3}$

$0 \le -\frac{2\pi}{3} \le \pi$



(makes sense b/c $\cos\theta$ is even)

d) $\cos(\cos^{-1}\pi)$ = undefined

$-1 \le \pi \le 1$?

False

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

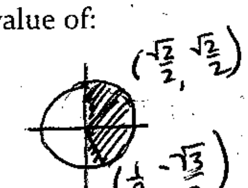
Example 9 Find the exact value of:

a) $\tan^{-1}1 = \theta$

$\tan\theta = 1$

$\tan\frac{\pi}{4} = 1$

$\tan^{-1}1 = \frac{\pi}{4}$



b) $\tan^{-1}(-\sqrt{3})$

$\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$\tan\theta = -\sqrt{3}$

$\tan-\frac{\pi}{3} = -\sqrt{3}$

$\tan^{-1}(-\sqrt{3}) = \frac{-\pi}{3}$

Example 10: Find the inverse function $f^{-1}$of $f(x) = 2\sin x - 1$. State the domain and the range of f and f^{-1}.

$y = 2\sin x - 1$

$x = 2\sin y - 1$ implicit def'n.

$x + 1 = 2\sin y$

$\frac{x+1}{2} = \sin y$

$\sin^{-1}\left(\frac{x+1}{2}\right) = \sin^{-1}(\sin y)$

$\sin^{-1}\left(\frac{x+1}{2}\right) = y$

$f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$

$-1 \leq \frac{x+1}{2} \leq 1$

$-2 \leq x+1 \leq 2$

$-3 \leq x \leq 1$

Domain/Range of f(x)

D: $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

R: $-3 \leq y \leq 1$

Domain and Range of $f^{-1}(x)$

D: $-3 \leq x \leq 1$

R: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Example 11: Solve the equation $\frac{3\sin^{-1} x}{3} = \frac{\pi}{3}$.

$\sin^{-1} x = \frac{\pi}{3}$

$\sin(\sin^{-1} x) = \sin\frac{\pi}{3}$

$x = \sin\frac{\pi}{3}$

$x = \frac{\sqrt{3}}{2}$