The Period of the Trigonometric Functions

Definition of periodic: A function is called periodic if there is a positive p such that whenever θ is in the domain of f, so is $\theta + p$ and $f(\theta + p) = f(\theta)$.

The smallest such p is called the fundamental period of f.

Periodic properties:

any multiple of 2π

$$\left.\begin{aligned} \sin\theta &= \sin(2\pi + \theta) \\ \cos\theta &= \cos(2\pi + \theta) \\ \csc\theta &= \csc(2\pi + \theta) \\ \sec\theta &= \sec(2\pi + \theta) \end{aligned}\right\} \text{Period of } 2\pi$$

multiple of π

$$\underbrace{\begin{aligned} \tan\theta &= \tan(\pi + \theta) \\ \cot\theta &= \cot(\pi + \theta) \end{aligned}}_{\text{Period of } \pi}$$

Example 1: Find the exact value without a calculator

a) $\sin\left(\frac{17\pi}{4}\right) = \sin\left(\frac{16\pi}{4} + \frac{\pi}{4}\right) = \sin\left(4\pi + \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$P = 2\pi$

b) $\cos(5\pi) = \cos(4\pi + \pi) = \cos(\pi) = -1$

$P = 2\pi$

c) $\tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{4\pi}{4} + \frac{\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1$

$P = \pi$

S	A
$-\ +$	$+\ +$
QII	QI
QIII	QIV
$-\ -$	$+\ -$
T	C

Example 2: If $\sin\theta < 0$ *and* $\cos\theta < 0$, in which Quadrant does θ lie?

Quadrant III

Derivation of some fundamental identities: use equation of unit circle

$(\sin\theta)^2 = \sin^2\theta$

$\neq \sin\theta^2$

$x^2 + y^2 = r^2$ $\quad x = \cos\theta$

$y^2 + x^2 = r^2$ $\quad y = \sin\theta$

$y^2 + x^2 = 1$

$\sin^2\theta + \cos^2\theta = 1$

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

Fundamental Identities: Memorized

$$\left.\begin{aligned} \tan\theta &= \frac{\sin\theta}{\cos\theta} \\ \cot\theta &= \frac{\cos\theta}{\sin\theta} \end{aligned}\right\} \text{Quotient identities}$$

$$\left.\begin{aligned} \csc\theta &= \frac{1}{\sin\theta} \\ \sec\theta &= \frac{1}{\cos\theta} \\ \cot\theta &= \frac{1}{\tan\theta} \end{aligned}\right\} \text{Reciprocal identities}$$

Pythagorean identities

$$\overbrace{\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \cot^2\theta = \csc^2\theta \qquad \tan^2\theta + 1 = \sec^2\theta}$$

Example 3: Given $\sin\theta = \frac{\sqrt{5}}{5}$ and $\cos\theta = \frac{2\sqrt{5}}{5}$, find the exact values of the four remaining trigonometric functions of θ using identities (no calc).

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{\cancel{\sqrt{5}}}{\cancel{5}} \cdot \frac{\cancel{5}}{2\cancel{\sqrt{5}}} = \boxed{\frac{1}{2}}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{5}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\cancel{5}\sqrt{5}}{\cancel{5}} = \boxed{\sqrt{5}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\frac{2\cancel{\sqrt{5}}}{\cancel{5}}}{\frac{\cancel{\sqrt{5}}}{\cancel{5}}} = 2$$

$$\cot\theta = \frac{1}{\tan\theta} = \boxed{2}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{2\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\cancel{5}\sqrt{5}}{2 \cdot \cancel{5}} = \boxed{\frac{\sqrt{5}}{2}}$$

Example 4: Find the exact value of each expression. Do not use a calculator.

a) $\tan 20° - \frac{\sin 20°}{\cos 20°}$

$= \tan 20° - \tan 20°$

$= 0$

b) $\sin^2\left(\frac{\pi}{12}\right) + \frac{1}{\sec^2\left(\frac{\pi}{12}\right)}$

$= \sin^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{\pi}{12}\right)$

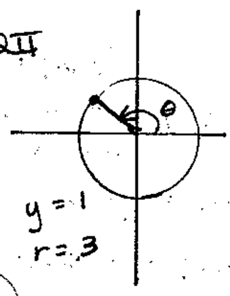
$= 1$

✱ Find the exact values of the trigonometric functions of an angle given one of the functions and the quadrant of the angle.

2 methods → circle, → identity

Example 5: Given $\sin\theta = \frac{1}{3}$ and $\cos\theta < 0$ find the exact value of each of the remaining five trigonometric functions.

Circle method QII



$\sin\theta = \frac{y}{r} = \frac{1}{3}$

$\cos\theta = \frac{x}{r} = \frac{-2\sqrt{2}}{3}$

$\tan\theta = \frac{y}{x} = \frac{1}{-2\sqrt{2}} = \frac{-\sqrt{2}}{4}$

$\sec\theta = \frac{r}{x} = \frac{-3}{2\sqrt{2}} = \frac{-3\sqrt{2}}{4}$

$\csc\theta = \frac{r}{y} = \frac{3}{1} = 3$

$\cot\theta = \frac{x}{y} = -2\sqrt{2}$

$y = 1$

$r = 3$

$x^2 + y^2 = r^2$

$x^2 + (1)^2 = (3)^2$

$x^2 + 1 = 9$

$x^2 = 8$

$x = \pm\sqrt{8}$

$x = -2\sqrt{2}$

identity method QII $\frac{\sin\theta}{\csc\theta}$ +

$\sin\theta = \frac{1}{3}$ $\quad \csc\theta = \frac{1}{\sin\theta} = 3$

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2\theta = 1 - \sin^2\theta$

$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$ discard +

$\cos\theta = -\sqrt{1 - (\frac{1}{3})^2} = -\sqrt{1 - \frac{1}{9}} = -\sqrt{\frac{8}{9}} = \frac{-\sqrt{8}}{3} = \frac{-2\sqrt{2}}{3}$

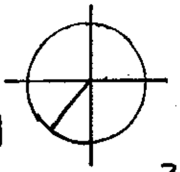
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{1}{\cancel{3}}}{\frac{-2\sqrt{2}}{\cancel{3}}} = \frac{-1}{2\sqrt{2}} = \frac{-\sqrt{2}}{4}$

$\cot\theta = \frac{1}{\tan\theta} = -2\sqrt{2}$

$\sec\theta = \frac{1}{\cos\theta} = \frac{-3}{2\sqrt{2}} = \frac{-3\sqrt{2}}{4}$

Example 6: Given that $\tan\theta = \frac{1}{2}$ and $\sin\theta < 0$, find the exact value of each of the remaining five trigonometric functions of θ.

circle method in QIII



$x = -2$

$y = -1$

$\tan\theta = \frac{y}{x} = \frac{1}{2}$

$\cot\theta = \frac{x}{y} = 2$

$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$

$\cos\theta = \frac{x}{r} = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$

$\csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-1} = -\sqrt{5}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{-2} = \frac{-\sqrt{5}}{2}$

$x^2+y^2=r^2$

$(-2)^2+(-1)^2=r^2$

$4+1=r^2$

$5=r^2$

$\sqrt{5}=r$

identity method QIII $\tan\theta$ + $\cot\theta$

$\tan\theta = \frac{1}{2}$ $\quad \cot\theta = \frac{1}{\tan\theta} = 2$

$\tan^2\theta + 1 = \sec^2\theta$

$-\sqrt{\tan^2\theta+1} = \sec\theta = -\sqrt{(\frac{1}{2})^2+1} = -\sqrt{\frac{5}{4}}$

$= \frac{-\sqrt{5}}{2}$

$\cos\theta = \frac{1}{\sec\theta} = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\quad \tan\theta\cos\theta = \sin\theta$

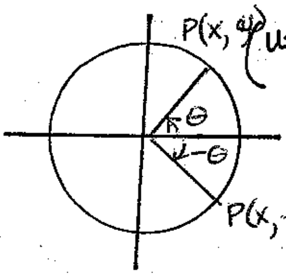
$(\frac{1}{2})(-\frac{2\sqrt{5}}{5}) = \sin\theta$

$-\frac{\sqrt{5}}{5} = \sin\theta$

$\csc\theta = \frac{1}{\sin\theta}$

$= \frac{-5}{\sqrt{5}} = -\sqrt{5}$

Even-Odd Properties of the Trigonometric Functions: Even? $f(x) = f(-x)$ graph is symmetric about y axis $\quad$ Odd? $f(-x) = -f(x)$ graph is symmetric about the origin



P(x, y) P(x, -y)

(use unit circle to determine properties)

$\sin\theta = y$ and $\sin(-\theta) = -y$ $\quad$ sine is odd

$-\sin\theta = -y \quad \therefore \boxed{\sin(-\theta) = -\sin\theta}$

$\cos\theta = x \quad \cos(-\theta) = x \quad \therefore \boxed{\cos\theta = \cos(-\theta)}$ cosine is even

$\boxed{\tan(-\theta)} = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin\theta}{\cos\theta} = \boxed{-\tan\theta}$ tangent is odd

$\csc(-\theta) = \frac{1}{\sin(-\theta)} = -\frac{1}{\sin\theta}$

$\csc(-\theta) = -\csc\theta$ (odd)

$\boxed{\cot(-\theta)} = \frac{\cos(-\theta)}{\sin(-\theta)} = \frac{\cos\theta}{-\sin\theta} = \boxed{-\cot\theta}$ cotangent is odd

$\boxed{\sec\theta = \sec(-\theta)}$ even

Example 7: Find the exact value of using even/odd properties.

a) $\sin(-45°)$

$= -\sin 45°$

$= \frac{-\sqrt{2}}{2}$

b) $\cos(-\pi)$

$= \cos\pi$

$= -1$

c) $\cot\left(\frac{-3\pi}{2}\right)$

$= -\cot\left(\frac{3\pi}{2}\right)$

$= -\frac{\cos(\frac{3\pi}{2})}{\sin(\frac{3\pi}{2})}$

$= \frac{-0}{-1} = 0$

d) $\tan\left(\frac{-37\pi}{4}\right)$

$= -\tan\left(\frac{37\pi}{4}\right)$

$= -\tan\left(\frac{36\pi}{4} + \frac{\pi}{4}\right)$

$= -\tan\left(9\pi + \frac{\pi}{4}\right)$

$= -\tan\left(\frac{\pi}{4}\right) = -1$