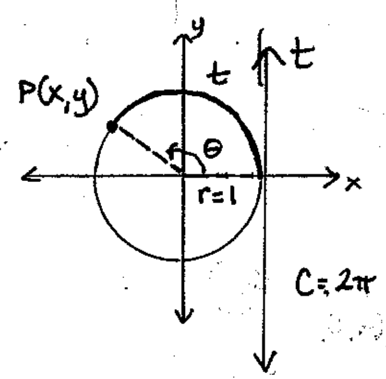
6.2 Trigonometric Functions: Unit Circle Approach



Definitions of the six trigonometric functions of t using the unit circle:

Let t be a real # and let $P=(x,y)$ be a point on the unit circle that corresponds to t.

Then $\sin t = y$ $\quad \csc t = \frac{1}{y}$ $\quad \cot t = \frac{x}{y}$

$\cos t = x$ $\quad \sec t = \frac{1}{x}$

$\tan t = \frac{y}{x}$

when x or $y = 0$ and is in the denominator, then the trig function is undefined.

Definition: If $t = \theta$ radians, then

$\sin t = \sin\theta$ $\quad \csc t = \csc\theta$ $\quad \tan t = \tan\theta$

$\cos t = \cos\theta$ $\quad \sec t = \sec\theta$ $\quad \cot t = \cot\theta$

(For examples 1 – 10 do not use a calculator)

Example 1: Let t be a real number and let $P = \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}\right)$ be the point on the unit circle that corresponds to t. Find the values of $\sin t$, $\cos t$, $\tan t$, $\csc t$, $\sec t$, and $\cot t$.

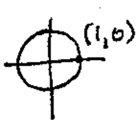
$\sin t = y = \frac{\sqrt{3}}{2}$ $\qquad \csc t = \frac{1}{y} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\cos t = x = \frac{-1}{2}$ $\qquad \sec t = \frac{1}{x} = -2$

$\tan t = \frac{y}{x} = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}} = -\sqrt{3}$ $\qquad \cot t = \frac{\frac{-1}{2}}{\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}$

Example 2: Find the $\boxed{\text{exact values}}$ of the six trigonometric functions of: –no calc

a) $\theta = 0 = 0°$

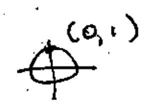


$\sin 0 = 0$ $\quad \csc 0 = \text{ud}$

$\cos 0 = 1$ $\quad \sec 0 = 1$

$\tan 0 = 0$ $\quad \cot 0 = \text{ud}$

b) $\theta = \frac{\pi}{2} = 90°$



$\sin\frac{\pi}{2} = 1$ $\quad \csc\frac{\pi}{2} = 1$

$\cos\frac{\pi}{2} = 0$ $\quad \sec\frac{\pi}{2} = \text{ud}$

$\tan\frac{\pi}{2} = \text{ud}$ $\quad \cot\frac{\pi}{2} = 0$

c) $\theta = \frac{3\pi}{2} = 270°$

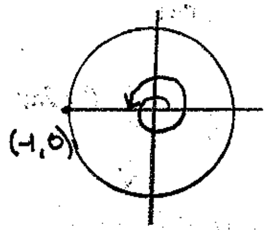
$\sin\frac{3\pi}{2} = -1$ $\quad \csc\frac{3\pi}{2} = -1$

$\cos\frac{3\pi}{2} = 0$ $\quad \sec\frac{3\pi}{2} = \text{ud}$

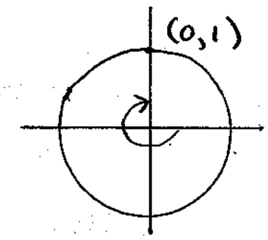
$\tan\frac{3\pi}{2} = \text{ud}$ $\quad \cot\frac{3\pi}{2} = 0$

Example 3: Find the exact value of

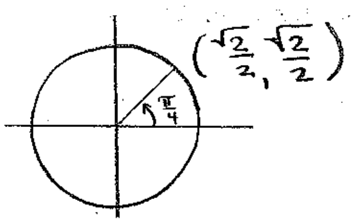
a) $\sin(3\pi) = 0$



b) $\cos(-270°) = 0$



Example 4: Find the exact values of the six trigonometric functions of $\frac{\pi}{4} = 45°$.



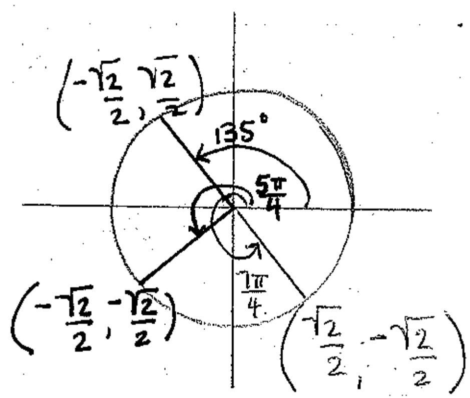
$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad$ $\csc\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad$ $\sec\frac{\pi}{4} = \sqrt{2}$

$\tan\frac{\pi}{4} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$ $\quad$ $\cot\frac{\pi}{4} = 1$

Example 5: Find the exact values of each expression. (Multiples of $\frac{\pi}{4}$)

a) $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$



b) $\sin 135° = \frac{\sqrt{2}}{2}$

c) $\tan\frac{7\pi}{4} = -1$

Example 6: Find the exact values of the six trigonometric functions of $\frac{\pi}{3} = 60°$.

$\cos\frac{\pi}{3} = \frac{1}{2}$

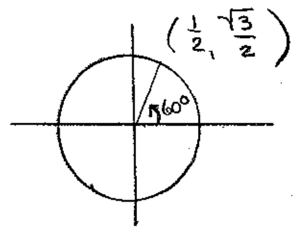
$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\tan\frac{\pi}{3} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$

$\sec\frac{\pi}{3} = 2$

$\csc\frac{\pi}{3} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

$\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



Example 7: Find the exact values of the six trigonometric functions of $\frac{\pi}{6} = 30°$.

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

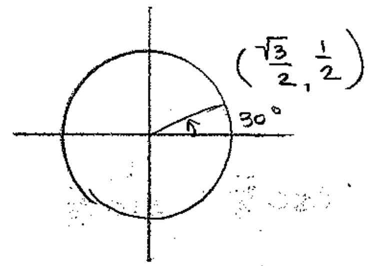
$\sin\frac{\pi}{6} = \frac{1}{2}$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

$\sec\frac{\pi}{2} = \frac{2\sqrt{3}}{3}$

$\csc\frac{\pi}{6} = 2$

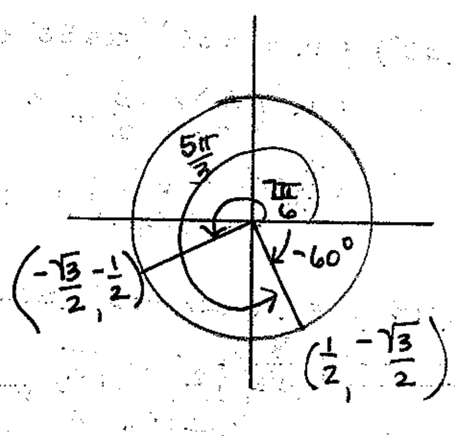
$\cot\frac{\pi}{6} = \sqrt{3}$



Example 8: Find the exact values of each expression.

a) $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

b) $\sin -60° = -\frac{\sqrt{3}}{2}$



c) $\tan\frac{5\pi}{3} = \dfrac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3}$

Example 9: Find the exact value of each expression.

a) $\sin 45^\circ \cos 180^\circ = \frac{\sqrt{2}}{2}\cdot -1 = \frac{-\sqrt{2}}{2}$

b) $\tan\frac{\pi}{4} - \sin\frac{3\pi}{2} = 1-(-1) = 2$

c) $2\csc\frac{\pi}{3} + \cot\frac{\pi}{4} = 2\cdot\frac{2}{\sqrt{3}} + 1 = \frac{4}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}} = \frac{(4+\sqrt{3})}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}} = \frac{4\sqrt{3}+3}{3}$

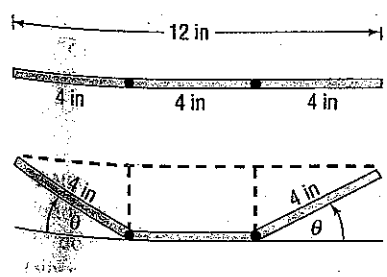
$\csc\frac{\pi}{3} = \frac{1}{\sin\frac{\pi}{3}}$ $\qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

Example 10: A rain gutter is to be constructed of aluminum sheets 12 inches wide. After marking off a length of 4 inches from each edge, this length is bent up at an angle θ. The area A of the opening may be expressed as function of θ as

$$A(\theta) = 16\sin\theta(\cos\theta + 1)$$

Find the area A of the opening for ① $\theta = 30°$, ② $\theta = 45°$, and ③ $\theta = 60°$.

Figure 29



① $A(30^\circ) = 16\sin 30^\circ(\cos 30^\circ + 1)$

$= 16\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2} + \frac{2}{2}\right)$

$= \cancel{8}^{4}\left(\frac{\sqrt{3}+2}{\cancel{2}_{1}}\right) = 4\sqrt{3} + 8 \approx 14.9 \text{ in}^2$

The area of the opening is about 14.9 in² when θ is 30°.

② $A(45^\circ) = 16\sin 45^\circ(\cos 45^\circ + 1)$

$= \cancel{16}^{8}\left(\frac{\sqrt{2}}{\cancel{2}_{1}}\right)\left(\frac{\sqrt{2}}{2} + \frac{2}{2}\right) = \cancel{8}^{4}\sqrt{2}\left(\frac{\sqrt{2}+2}{\cancel{2}_{1}}\right) = 4\cdot 2 + 8\sqrt{2} = 8 + 8\sqrt{2}$

$\approx 19.3 \text{ in}^2$

The area of the opening is about 19.3 in² when θ is 45°.

③ $A(60^\circ) = 16\sin 60^\circ(\cos 60^\circ + 1)$

$= \cancel{16}^{8}\cdot\frac{\sqrt{3}}{\cancel{2}_{1}}\left(\frac{1}{2} + 1\right)$

$= \cancel{8}^{4}\sqrt{3}\left(\frac{3}{\cancel{2}_{1}}\right) = 12\sqrt{3} \approx$

20.8 in^2

The area of the opening is about 20.8 in² when θ is 60°.

$$\frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}}$$

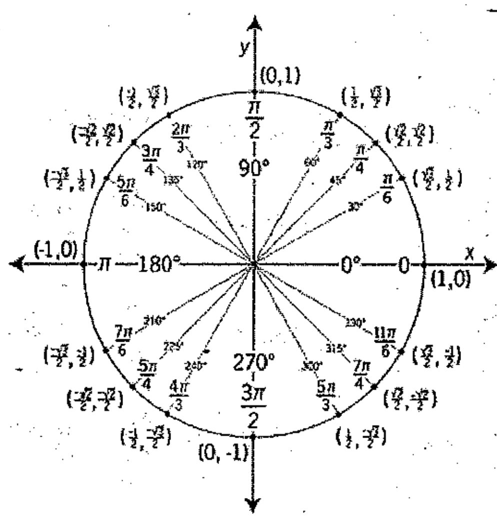
6.2 Summary table

y x $\frac{y}{x} = \frac{\sin\theta}{\cos\theta}$ $\frac{1}{y} = \frac{1}{\sin\theta}$ $\frac{x}{y} = \frac{\cos\theta}{\sin\theta}$

θ (radians)	θ (degrees)	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0	$0°$	0	1	0	undefined	1	undefined
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{2}$	$90°$	1	0	undefined	1	undefined	0

$$\tan\theta = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \qquad \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

6.3 Properties of Trigonometric Functions and some Fundamental Identities



properties are not dependent on the radius, we will use the unit to establish all properties.

	Function	Symbol	Domain $\theta \to$ input (t)	Range output
y	sine	$\sin\theta$	$\mathbb{R}$	$-1 \le \sin\theta \le 1$
x	cosine	$\cos\theta$	$\mathbb{R}$	$-1 \le \cos\theta \le 1$
$\frac{y}{x}$	tangent	$\tan\theta$	$\mathbb{R}$ except odd integer multiples of $\frac{\pi}{2}$.	$y \to 1$ and $x \to 0$ $\frac{0}{1}$ start and tan blowsup $\mathbb{R}$
$\frac{1}{y}$	cosecant	$\csc\theta$	$\mathbb{R}$ except integer multiples of π.	$\csc\theta \le -1$ or $\csc\theta \ge 1$
$\frac{1}{x}$	secant	$\sec\theta$	$\mathbb{R}$ except odd integer multiples of $\frac{\pi}{2}$	$\sec\theta \le -1$ or $\sec\theta \ge 1$
$\frac{x}{y}$	cotangent	$\cot\theta$	$\mathbb{R}$ except integer multiples of π	$\mathbb{R}$