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5.8 Exponential Growth and Decay Models; Newton's Law; Logistic Growth and Decay Models

The exponential law

$$A(t) = A_0 e^{kt}$$

The law of uninhibited growth or decay.

t = time

A_0 = initial amount

$k \to$ constant specific to situation

$k > 0$ growth rate (exponential growth)

$k < 0$ decay rate (exponential decay)

Uninhibited Growth of Cells:

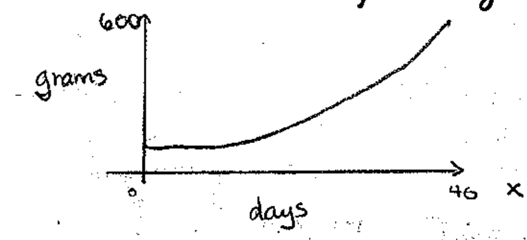
$$N(t) = N_0 e^{kt}$$

Example 1: A colony of bacteria grows according to the law of uninhibited growth according to the function $N(t) = 100e^{0.045t}$, where N is measured in grams and t is measured in days.

(a) Determine the initial amount of bacteria. Initially the population weighed 100 grams.

(b) What is the growth rate of the bacteria? $k = .045$ $\to$ 4.5 % per day

(c) Graph the function using a graphing utility.



(d) What is the population after 5 days?

$N(5) = 100e^{.045(5)}$

$N(5) \approx 125.23$

After 5 days the population will weigh about 125.23 grams.

(e) How long will it take for the population to reach 140 grams?

$$\frac{140}{100} = \frac{100e^{.045t}}{100}$$

$$1.4 = e^{.045t}$$

$$\ln(1.4) = .045t$$

$$\frac{\ln(1.4)}{.045} = t \approx 7.48$$

The population will reach 140 grams after approximately $7\frac{1}{2}$ days.

(f) What is the doubling time for the population? Initial $\to$ 100 double $\to$ 200

$$200 = 100e^{.045t}$$

$$2 = e^{.045t}$$

$$\ln 2 = .045t$$

$$\frac{\ln 2}{.045} = t \approx 15.40$$

The population will double after about 15 days, 9hrs and 36 minutes.

$.4 \text{ days} \cdot \frac{24 \text{ hrs}}{1 \text{ day}} = 9.6 \text{ hrs.}$

Example 2: A colony of bacteria grows according to the law of uninhibited growth.
(a) If N is the number of cells and t is the time in hours, express N as a function of t.

$$N(t) = N_0 e^{kt}$$

(b) If the number of bacteria doubles in 3 hours, find the function that gives the number of cells in the culture. (find k)

$N(3) = 2N_0$
$t = 3$

$$\frac{2N_0}{N_0} = \frac{N_0 e^{k \cdot 3}}{N_0}$$

$$2 = e^{3k}$$

$$\ln 2 = 3k$$

$$\frac{\ln 2}{3} = k \quad \text{use this}$$

check if k is reasonable..
$k \approx .231$ 23.1% ✓

$$N(t) = N_0 e^{\frac{\ln 2}{3} t}$$

$$N(t) = N_0 e^{\frac{t \ln 2}{3}}$$

(c) How long will it take for the size of the colony to triple?

$N(t) = 3N_0$

$$\frac{3\cancel{N_0}}{\cancel{N_0}} = \frac{\cancel{N_0} e^{\left(\frac{\ln 2}{3}\right)t}}{\cancel{N_0}}$$

$$3 = e^{\left(\frac{\ln 2}{3}\right)t}$$

$$\ln 3 = \left(\frac{\ln 2}{3}\right)t \quad \text{multiply by reciprocal}$$

$$\frac{3 \ln 3}{\ln 2} = t$$

$$\approx 4.75$$

It will take about 4 hours and 45 minutes for the colony size to triple.

(d) How long will it take for the population to double a second time (that is, increase four times)?

If it takes 3 hrs. to double the first time, it will 3hrs to double the 2nd time. Thus it will 6 hours to increase 4 times.

Uninhibited Radioactive Decay –

$$A(t) = A_0 e^{kt}$$

$k < 0$
decay factor

each radioactive substance has its own unique $\frac{1}{2}$ life.
(the time it takes for $\frac{1}{2}$ of the radioactive material to decay)

C_{12} - stable isotope } both in living organisms
C_{14} - radioactive

Ratio $C_{12}:C_{14}$ remains constant as long as organism.

Example 3: Traces of burned wood along with ancient stone tools in an archeological dig in Chile were found to contain approximately 1.67% of the original amount of carbon 14.

(a) If the half-life of carbon 14 is 5600 years, approximately when was the tree cut and burned?

① $A(t) = A_0 e^{kt}$

$$\frac{\frac{1}{2}A_0}{A_0} = \frac{A_0 e^{k \cdot 5600}}{A_0}$$

$$\frac{1}{2} = e^{5600k}$$

$$\ln(.5) = 5600k$$

$$\frac{\ln(.5)}{5600} = k$$

$k \approx -.000124$

$.0124\%$ per year

② $.0167A_0 = A_0 e^{\frac{\ln(.5)}{5600} \cdot t}$

$$.0167 = e^{\frac{\ln .5}{5600} t}$$

$$\ln(.0167) = \frac{\ln(.5)}{5600} t$$

$$\frac{5600 \ln(.0167)}{\ln(.5)} = t \approx 33{,}062 \text{ years}$$

The tree died about 33,000 years.

sketchy sketch (b) Using a graphing utility, graph the relation between the percentage of carbon 14 remaining and time.

x

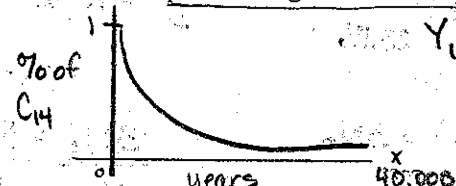
$Y_1 = e^{\frac{\ln .5}{5600} x}$

$X_{min} = 0$

$X_{max} = 40{,}000$

$Y_{min} = 0$

$Y_{max} = 1$



(c) Determine the time that elapses until half of the carbon 14 remains. This answer should equal the half-life of carbon 14. verify model on calc.

Y_1 = same as above

$Y_2 = .5$

intersect $x = 5600$ $Y = .5$

(d) Use a graphing utility to verify the answer found in part (a).

$Y_1 =$ "

$Y_2 = .0167$

intersect $(33062.45, .0167)$

Newton's Law of Cooling –

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Example 5: Fruit flies are placed in a half-pint milk bottle with a banana (for food) and yeast plants (for food and to provide a stimulus to lay eggs). Suppose that the fruit fly population after t days is given by

$$P(t) = \frac{230}{1+56.5e^{-0.37t}}$$

a) State the carrying capacity and the growth rate.

The carrying capacity is 230 fruit flies. The growth rate is 37% per day.

b) Determine the initial population.

$$P(0) = \frac{230}{1+56.5e^{-.37(0)}} = \frac{230}{57.5} = 4$$

Initially there were 4 fruit flies in the bottle.

c) What is the population after 5 days?

$$P(5) = \frac{230}{1+56.5e^{-.37(5)}}$$

$$P(5) \approx 23.27$$

There should be about 23 fruit flies after 5 days.

d) How long does it take for the population to reach 180?

$$(1+56.5e^{-.37t})\frac{180}{180} = \frac{230}{\cancel{1+56.5e^{-.37t}}} \cdot (\cancel{1+56.5e^{-.37t}})$$

& then ÷ by 180

$$1+56.5e^{-.37t} = \frac{230}{180}$$

$$1+56.5e^{-.37t} = \frac{23}{18} - \frac{18}{18}$$

$$-1$$

$$56.5e^{-.37t} = \frac{5}{18}$$

ASIDE

$56.5 = 56\frac{1}{2}$

$\frac{113}{2}$

$$e^{-.37t} = \frac{5}{\cancel{18}\,9} \cdot \frac{\cancel{2}\,1}{113}$$

$$\ln e^{-.37t} = \ln\frac{5}{1017}$$

$$-.37t = \ln\left(\frac{5}{1017}\right)$$

$$t = \frac{\ln\left(\frac{5}{1017}\right)}{-.37}$$

$$t \approx 14.365 \text{ days}$$

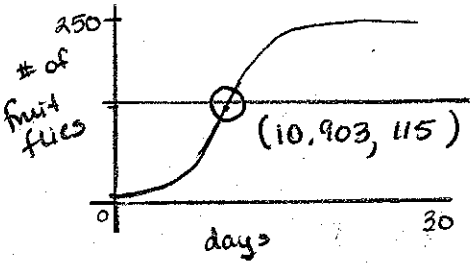
$$.365 \text{ day} \cdot \frac{24h}{1 \text{ day}} = 8.76 \text{ hr}$$

It will take approximately 14 days, 8 hrs and 45 minutes for the population to reach 180 fruit flies.

e) Use a graphing calculator to determine how long it takes for the population to reach one-half of the carrying capacity.

$$Y_1 = \frac{230}{1+56.5e^{-.37t}}$$

$$\frac{1}{2}(230) \quad Y_2 = 115$$



It will take about 10 days, 21 hours and 40 minutes for the population to reach ½ of the carrying capacity.

Logistic Model – exponential model with dampening

$$P(t) = \frac{c}{1+ae^{-bt}}$$

a, b, c are constants

$c > 0$

growth model when $b > 0$ → growth rate

decay model when $b < 0$ decay rate

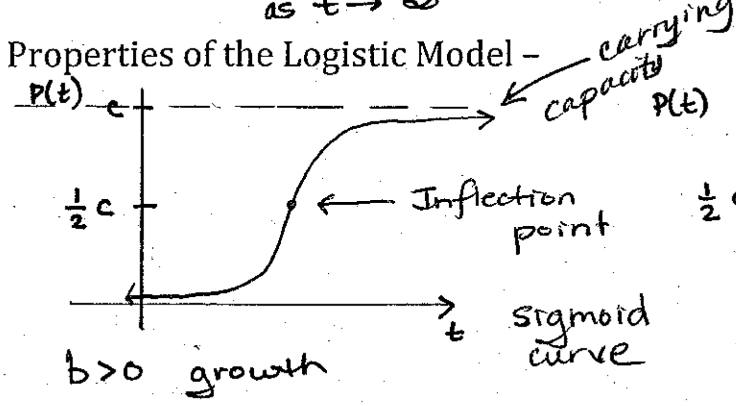
in growth models

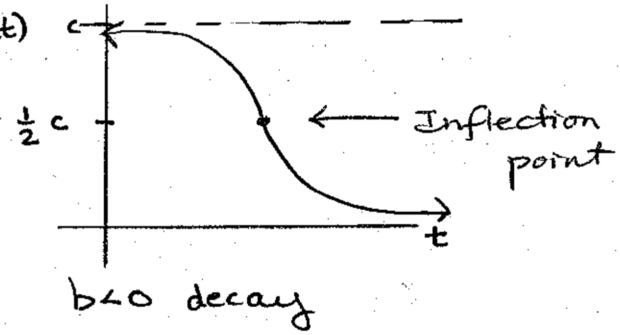
c = carrying capacity

$$\lim_{t \to \infty} P(t) = c$$

b/c $e^{-bt} \to 0$ as $t \to \infty$

Properties of the Logistic Model –





1. D: $\mathbb{R}$ R: $\{y \mid 0 < y < c\}$
2. No x-intercepts, y int. $(0, P(0))$
3. 2 horizontal asymptotes $y = 0$ $y = c$
   No vertical asymptotes
4. Increasing if $b > 0$ } Means that it is 1 to 1.
   Decreasing if $b < 0$ } so it has an inverse.
5. Has inflection point where graph changes concavity $P(t) = \frac{1}{2}c$
6. The graph is smooth and continuous.

Example 6: The EFISCEN wood product model classifies wood products according to their life-span. There are four classifications: short (1 year), medium short (4years) medium long (16 years), and long (50 years). Based on data obtained from the European Forest Institute, the percentage of remaining wood products after t years for wood products with long life-spans (such as those used in the building industry) is given by

$$P(t) = \frac{100.3952}{1 + 0.0316e^{0.0581t}}$$

a) What is the decay rate?

$b = -.0581$ The decay rate is 5.81% per year.

b) What is the percentage of remaining wood products after 10 years?

$$P(10) = \frac{100.3952}{1+.0316e^{.0581(10)}}$$

$$P(10) \approx 95.027$$

About 95% of the wood products remain after 10 years.

c) How long does it take for the percentage of remaining wood products to reach 50%?

$$50 = \frac{100.3952}{1+.0316e^{.0581t}}$$

$$50(1+.0316e^{.0581t}) = 100.3952$$

$$1+.0316e^{.0581t} = \frac{100.3952}{50}$$

$$1+.0316e^{.0581t} = 2.007904$$

$$.0316e^{.0581t} = 1.007904$$

$$e^{.0581t} = \frac{1.007904}{.0316}$$

$$e^{.0581t} = \frac{62994}{1975}$$

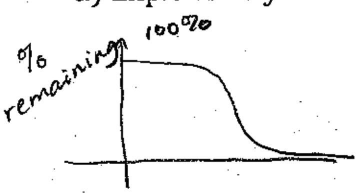
$$.0581t = \ln\left(\frac{62994}{1975}\right)$$

$$t = \frac{\ln\left(\frac{62994}{1975}\right)}{.0581}$$

$$t \approx 59.595 \text{ yrs.}$$

The model predicts that it will take a little over $59\frac{1}{2}$ years for the percentage of remaining wood products to reach 50%.

d) Explain why the numerator given in the model is reasonable.



The ~~numerator~~ of 100.3952 is reasonable because it is close to 100 which we know is the initial percent of wood products remaining.