$y=a^x$
$x=a^y$ implicit def'n
$\log_a x = y$ explicit def'n.

5.4 Logarithmic Functions
Definition of a logarithmic function –

The logarithm function to the base a, where $a>0$ and $a \neq 1$ is denoted $y=\log_a x$.

$y=\log_a x$ iff $x=a^y$
the argument

Example 1: Relating logarithms to Exponents

(a) If $= \log_3 x$, then
$y=\log_3 x$ then $x=3^y$

(b) If $= \log_5 x$, then
$y=\log_5 x$ then $x=5^y$

Example 2: Change each exponential statement to an equivalent statement involving a logarithm.

(a) $1.2^3 = m$ (b) $e^b = 9$ (c) $a^4 = 24$

$\log_{1.2} m = 3$ $\log_e 9 = b$ $\log_a 24 = 4$

Example 3: Change each logarithmic statement to an equivalent statement involving an exponent.

(a) $\log_a 4 = 5$ (b) $\log_e b = -3$ (c) $\log_3 5 = c$

$4 = a^5$ $b = e^{-3}$ $5 = 3^c$

use def'n of log and $a^u = a^v$ then $u = v$

Example 4: Find the exact value of ...

(a) $\log_2 16 =$

$2^x = 16$

$2^x = 2^4 \quad x = 4$

(b) $\log_3 \frac{1}{27} =$

$3^x = \frac{1}{27}$

$3^x = 3^{-3} \quad x = -3$

Domain and Range of Logarithmic Functions

Domain of log function: $\{x \mid x > 0\}$

(Range of exponential function: $\{y \mid y > 0\}$

Range of log function: $\{y \mid -\infty < y < \infty\}$ $\mathbb{R}$

(Domain of exponential: $\mathbb{R}\}$

Example 5: Find the domain of each logarithmic function.

(a) $F(x) = \log_2(x+3)$

$x + 3 > 0$

$x > -3$

D of $F(x)$

$\{x \mid x > -3\}$

(b) $G(x) = \log_5\left(\frac{1+x}{1-x}\right)$

$\frac{1+x}{1-x} > 0$ D of $G(x)$:

$\{x \mid -1 < x < 1\}$

$-\quad + \quad -$

$-2 \quad -1 \quad 0 \quad 1 \quad 2$

$\frac{-1}{3} \quad 1 \quad \frac{3}{-1} = -3$

(c) $h(x) = \log_{\frac{1}{2}}|x|$

$|x| > 0$

$\{x \mid x \neq 0\}$

$|x| > 0$

$x > 0$ or $x < 0$

$|2x+1| < 1$

$2x+1 < 1$ and $2x+1 > -1$

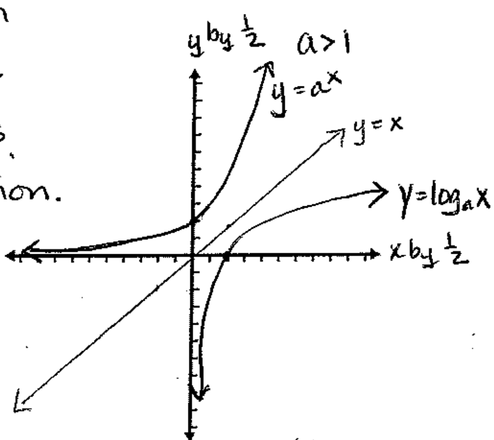
$2x < 0$ and $2x > -2$

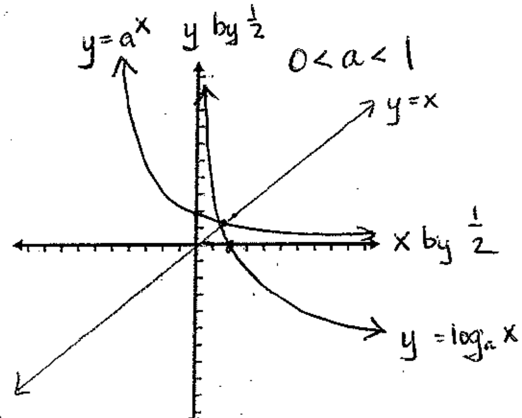
$x < 0$ and $x > -1$

$-1 < x < 0$

Properties of the Logarithmic Function $f(x) = \log_a x$

1. D: $\{x \mid x > 0\}$ R: $\mathbb{R}$
2. No y-int., x-int. (1,0)
3. No horizontal asymptote, vertical asymptote $x = 0$
4. If $a > 1$, log is an increasing function.
   If $0 < a < 1$, log is a decreasing function.
   Log's are 1-to-1.





5. Points on graph
   $(1,0)$, $(\frac{1}{a}, -1)$,
   $(a, 1)$
6. Graph is smooth and continuous.

Base 10 (Common log)

$y = \log x$ iff $x = 10^y$

Base e (Natural log)

$y = \ln x$ iff $x = e^y$

Example 7: (a) Find the domain of the logarithmic function $f(x) = 3\log(x-1)$
(b) Graph f
(c) From the graph, determine the range and vertical asymptote of f.
(d) Find f^{-1}, the inverse of f.
(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.
(f) Graph f^{-1}

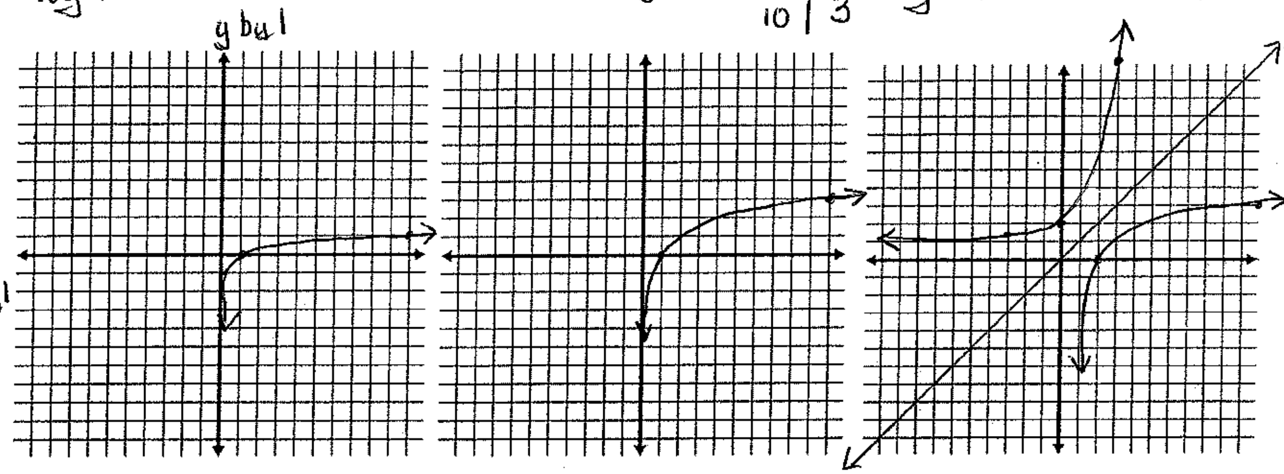
a) $x - 1 > 0$ D: $\{x \mid x > 1\}$

$x > 1$

$f(x) = \log x \xrightarrow{\text{vertical stretch}} f(x) = 3\log x \xrightarrow[\text{right 1}]{\text{shift}} f(x) = 3\log(x-1)$

x	f(x)
$\frac{1}{10}$	-1
1	0
10	1

x	f(x)
$\frac{1}{10}$	-3
1	0
10	3



x	f(x)
$1\frac{1}{10}$	-3
2	0
11	3

c) R: $\mathbb{R}$

V.A.: $x = 1$

d) $y = 3\log(x-1)$

$x = 3\log(y-1)$ implicit

$\frac{x}{3} = \log(y-1)$

$10^{\frac{x}{3}} = y - 1$

$10^{\frac{x}{3}} + 1 = y$

$f^{-1}(x) = 10^{\frac{x}{3}} + 1$ explicit

e) $f^{-1}(x)$ D: $\mathbb{R}$

R: $\{y \mid y > 1\}$

Example 6: (a) Find the domain of the logarithmic function $f(x) = -\ln(x-2)$ ← algebraically

(b) Graph f ⟶ using transformations.

(c) From the graph, determine the range and vertical asymptote of f.

(d) Find f^{-1}, the inverse of f. ⟶ algebraically

(e) Use f^{-1} to confirm the range of f found in part (c). From the domain of f, find the range of f^{-1}.

(f) Graph f^{-1}

a) $x-2>0$ $\quad$ D: $\{x \mid x>2\}$

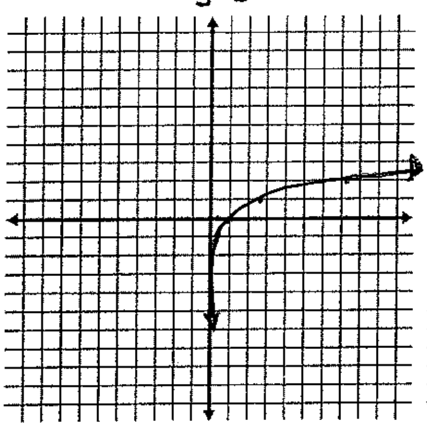
$\quad +2 \quad +2$

$x>2$

$f(x)=\ln x \xrightarrow[\text{the x-axis}]{\text{reflection over}} f(x)=-\ln x \xrightarrow[\text{right 2}]{\text{shift}} f(x)=-\ln(x-2)$

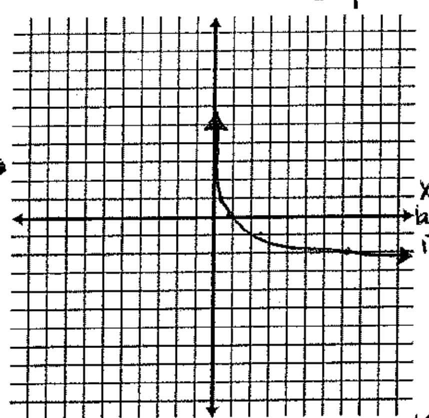
x	f(x)
.37 $\frac{1}{e}$	-1
1	0
2.72 e	1
7.38 e^2	2

x by 1, y by 1



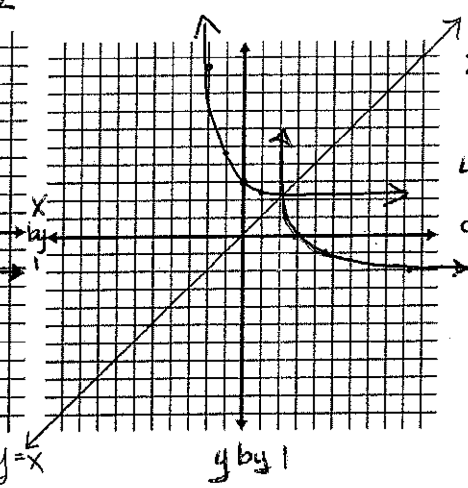
x	f(x)
.37	1
1	0
2.72	-1
7.38	-2

x by 1, y by 1



x	f(x)
2.37	1
3	0
4.72	-1
9.38	-2

y=x, y by 1



c) R: $\mathbb{R}$

V.A.: $x=2$

d) $y=-\ln(x-2)$

implicit $\dfrac{x}{-1}=\dfrac{-\ln(y-2)}{-1}$

$-x=\ln(y-2)$

$e^{-x}=y-2$

$e^{-x}+2=y$

explicit $f^{-1}(x)=e^{-x}+2$

e) $f^{-1}(x)$ D: $\mathbb{R}$

R: $\{y \mid y>2\}$

Example 8: Solving logarithmic equations

Note: B/c domain of log's is restricted, easy to get extraneous solutions — so always check your solutions

(a) $\log_3(4x-7)=2$

$4x-7=3^2$

$4x-7=9$

$4x=16$

$\boxed{x=4}$

check

$4 \cdot 4-7=9$

positive argument solution exists.

(b) $\log_x 64=2$

$64=x^2$

$\sqrt{x^2}=\sqrt{64}$

$\sqrt{x^2}=\pm\sqrt{64}$

$x=\pm 8$

x is the base

bases must be +

so −8 is extraneous

$\boxed{x=8}$

Example 9: Using logarithms to solve exponential equations

Solve $e^{2x}=5$

$2x=\ln 5$

$x=\frac{\ln 5}{2}$ exact answer

$x\approx .805$ approximation

Example 10: The blood alcohol concentration (BAC) is the amount of alcohol in a person's bloodstream. A BAC of 0.04% means that a person has 4 parts alcohol per 10,000 parts blood in the body. Relative risk is defined as the likelihood of one event occurring divided by the likelihood of a second event occurring. For example, if an individual with a BAC of 0.02% is 1.4 times as likely to have a car accident as an individual who has not been drinking, their relative risk of an accident with a BAC of 0.02% is 1.4. Recent medical research suggest that the relative risk R of having an accident while driving a car can be modeled by the equation

$$R = e^{kx}$$

where x is the percent of concentration of alcohol in the bloodstream and k is a constant.

(a) Research indicates that the relative risk of a person having an accident with a BAC of 0.02% is 1.4. Find the constant k in the equation.

(b) Using this value of k, what is the relative risk if the concentration is 0.17%?

(c) Using this same value of k, what is the BAC corresponds to a relative risk of 100?

(d) If the law asserts that anyone with a relative risk of 5 or more should not have driving privileges, at what concentration of alcohol in the bloodstream should a driver be arrested and charged with a DUI (driving under the influence)?

a) $R = e^{kx}$

$1.4 = e^{k(.02)}$

$\frac{.02k}{.02} = \frac{\ln(1.4)}{.02}$

$k = \frac{\ln 1.4}{.02}$ exact (use)

$k \approx 16.82$ approximation

b) $R = e^{\left(\frac{\ln 1.4}{.02}\right)(.17)}$

$R \approx 17.46$

An individual with a BAC of .17% is 17.46 times more likely to get into an accident.

c) $100 = e^{\left(\frac{\ln 1.4}{.02}\right)x}$

$\left(\frac{\ln 1.4}{.02}\right)x = \ln 100$

$x = \frac{.02 \ln 100}{\ln 1.4}$

$x \approx .27$

An individual with a BAC of .27% is 100 times more likely to get into a car accident.

d) $5 = e^{\left(\frac{\ln 1.4}{.02}\right)x}$

$\ln 5 = \frac{\ln 1.4}{.02}(x)$

$\frac{.02 \ln 5}{\ln 1.4} = x$

$.096 \approx x$

A driver with a BAC of .096% or greater should be arrested if a relative risk of 5 is unacceptable.