5.3 Exponential functions

Exponential function

$f(x) = Ca^{x}$ ← exp.

↳ base

variable = exponent

Power function

Power $\{ f(x) = Cx^{a}$ ← exponent

↖ base

variable = base

Example 1: Using a calculator to evaluate powers of 2
Evaluate

a) $2^{1.4}$ b) $2^{1.41}$ c) $2^{1.414}$ d) $2^{1.4142}$ e) $2^{\sqrt{2}}$

≈ 2.639015822 ≈ 2.657371628 ≈ 2.66474965 ≈ 2.665119089 $\approx 2.66514414 3$

An exponential function is a function of the form... $f(x) = Ca^{x}$ x can be any real #

a is the base

a is positive (b/c of fractional exponents)

$(a>0, a\neq 1)$

$C \Rightarrow$ initial value (y-int on graph)

a = growth/decay factor $a>1$ exp. growth $0<a<1$ exp. decay.

Laws of Exponents –

If s, t, a, b are real #'s, and $a>0$ and $b>0$ then

$$a^{s}\cdot a^{t} = a^{s+t} \qquad (a^{s})^{t} = a^{s\cdot t} \qquad (ab)^{s} = a^{s}b^{s} \qquad \left(\frac{a}{b}\right)^{s} = \frac{a^{s}}{b^{s}}$$

$$1^{s} = 1 \qquad a^{0} = 1 \qquad a^{-s} = \frac{1}{a^{s}} \qquad \frac{1}{a^{-s}} = a^{s} \qquad a^{\frac{1}{s}} = \sqrt[s]{a}$$

$$a^{\frac{t}{s}} = \sqrt[s]{a^{t}} = \underbrace{\left(\sqrt[s]{a}\right)^{t}}_{\text{No calc}}$$

$x_2 > x_1$.

To determine if a function is linear or exponential or neither...

Use below rules if x's increasing by 1.

Is it linear?

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_2 - y_1}{1} = y_2 - y_1$$

the $\boxed{\text{difference}}$ b/t consecutive y's is constant and $= m$ ← arithmetic sequences/series

$y = mx + b$ ⇐ write.

Is it exponential?

$\frac{y_2}{y_1}$ is constant → the $\boxed{\text{ratio}}$ b/t consecutive y's is constant, then you have exponential $a(\text{base}) = \frac{y_2}{y_1}$

geometric seq/series →

$y = Ca^{x}$ ⇐ write

Theorem: For an exponential funtion $f(x) = Ca^x$

$$\frac{f(x+1)}{f(x)} = a \qquad f(x+1) = af(x)$$

Example 2: Determine whether the given function is linear, exponential, or neither. For those that are linear, find a linear function that models the data. For those that are exponential, find an exponential function that models the data.

Ask are x's ↑ by 1?

a)

x	y
-1	5
0	2
1	-1
2	-4
3	-7

↑by 1

-3, -3, -3, -3

guess linear

$y_2 - y_1$

$2 - 5 = -3$
$-1 - 2 = -3$
$-4 + (+1) = -3$
$-7 + (+4) = -3$

$m = -3$
constant must be linear

$y = mx + b$

$y = -3x + 2$

b)

x	y
-1	2
0	4
1	7
2	11
3	15

✓

guess not linear .... not exp.

$y_2 - y_1$

$4 - 2 = 2$
$7 - 4 = 3$

$\frac{y_2}{y_1}$

$\frac{4}{2} = 2$

$\frac{7}{4} = \frac{7}{4}$

Neither constant, so neither function.

c)

x	y
-1	32
0	16
1	8
2	4
3	2

x's go up by 1

guess exponential

$y_2 - y_1$

$16 - 32 = -16$
$8 - 16 = -8$
Not constant

Not linear

$\frac{y_2}{y_1}$

$\frac{16}{32} = \frac{1}{2}$
$\frac{8}{16} = \frac{1}{2}$
$\frac{4}{8} = \frac{1}{2}$
$\frac{2}{4} = \frac{1}{2}$

Constant ratio - exponential

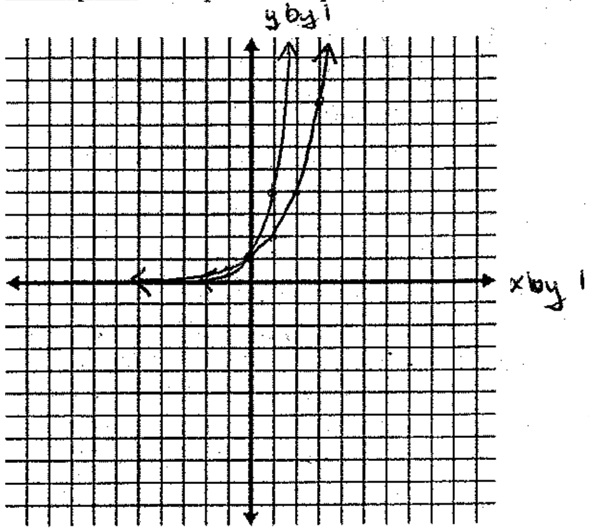
$y = Ca^x$

$a = \frac{1}{2}$

$C = 16$

$y = 16\left(\frac{1}{2}\right)^x$

Example 3: Graph the exponential function: $f(x) = 2^x$



x	f(x)
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

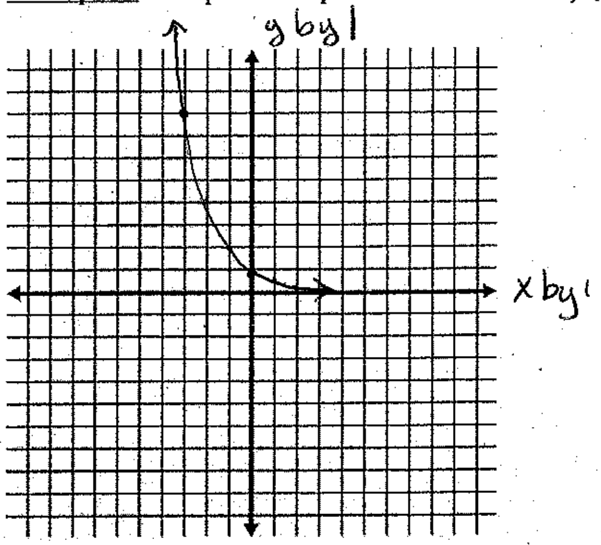
$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$

$f(x) = 4^x$

x	f(x)
−1	$\frac{1}{4}$
0	1
1	4
2	16

flip over the y-axis.

Example 4: Graph the exponential function: $f(x) = \left(\frac{1}{2}\right)^x = 2^{-x}$

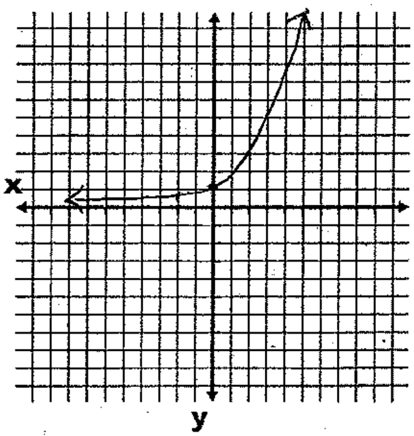


x	f(x)
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

$\left(\frac{1}{2}\right)^{-2} = \left(\frac{2}{1}\right)^2$

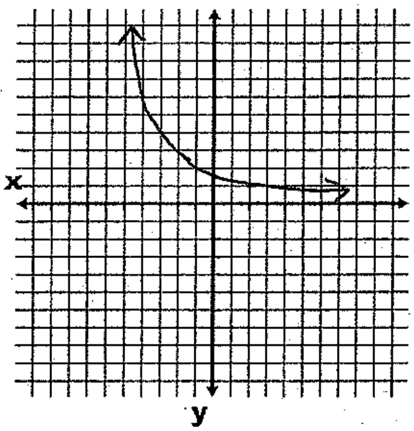
$= 4$

Properties of the Exponential Function $f(x) = a^x$, $a > 1$ Exponential growth



1. Domain $\mathbb{R}$ R: $\{y \mid y > 0\}$
2. y-int. $(0,1)$, no x-intercepts
3. Horiz. asym. $y = 0$ as $x \to -\infty$
   unbounded as $x \to +\infty$
   No vertical asym.
4. Increasing over entire domain $\therefore$ it is 1-to-1
5. The graph contains $(0,1), (1,a), (-1, \frac{1}{a})$
6. The graph is smooth and continuous.

Properties of the Exponential Function $f(x) = a^x$, $0 < a < 1$ Exponential decay

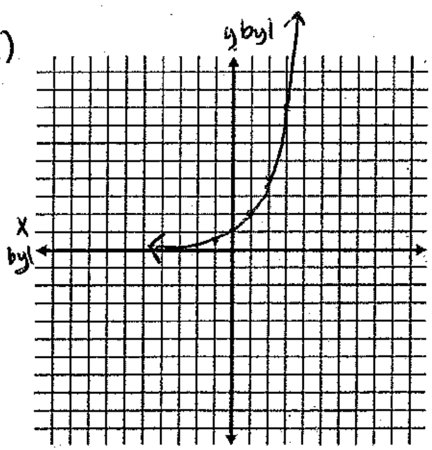


1. Domain $\mathbb{R}$ R: $\{y \mid y > 0\}$
2. y-int. $(0,1)$, no x-int.
3. Horizontal asy. $y = 0$ as $x \to +\infty$
   unbounded as $x \to -\infty$
   No vert. asy.
4. Decreasing over entire domain $\therefore$ it is 1-to-1
5. Contains $(0,1), (1,a), (-1, \frac{1}{a})$
6. Graph is smooth

Example 5: Graph $f(x) = 2^{-x} - 3$ and determine the domain, range, and horizontal asymptote of f.

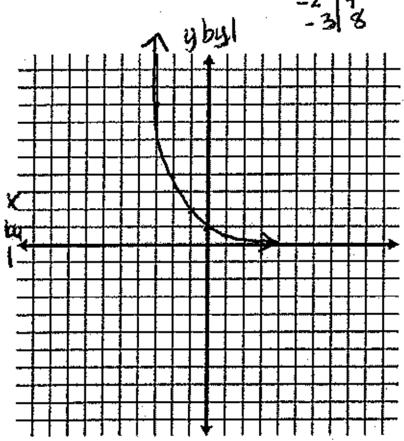
$f(x) = 2^x$

x	f(x)
0	1
1	2
-1	$\frac{1}{2}$
2	4
3	8



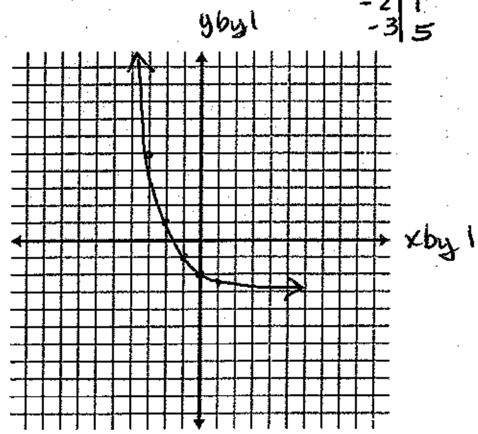
$f(x) = 2^{-x}$

x	f(x)
0	1
-1	2
1	$\frac{1}{2}$
-2	4
-3	8



$f(x) = 2^{-x} - 3$

x	f(x)
0	-2
-1	-1
1	$-2\frac{1}{2}$
-2	1
-3	5



D: $\mathbb{R}$
R: $\{y \mid y > -3\}$
Hor. Asp.: $y = -3$

one definition

The number e is defined .... as the number the expression

$$\left(1+\frac{1}{n}\right)^n \text{ approaches as } n \to \infty$$

$f(x) = e^x$

<u>The</u> exponential function $\quad e \approx 2.718281 28 \ldots$

<u>Example 6</u>: Graph $f(x) = -e^{x-3}$ and determine the domain, range, and horizontal asymptote of f.

$f(x) = e^x$

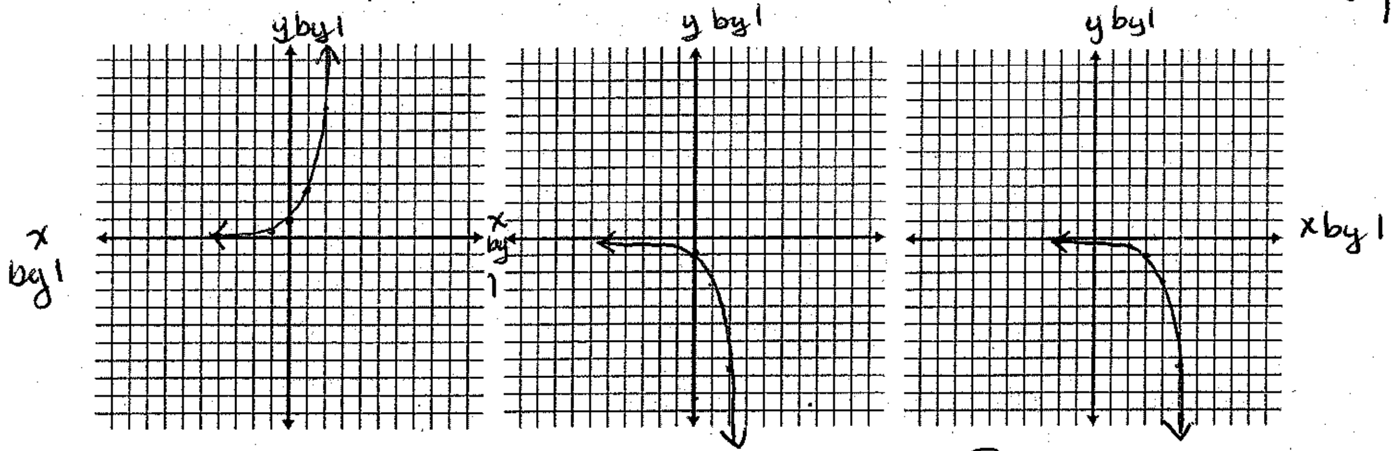
x	f(x)
-1	$\frac{1}{e} \approx .37$
0	1
1	$e \approx 2.72$
2	$e^2 \approx 7.38$

$\longrightarrow f(x) = -e^x$

x	f(x)
-1	-.37
0	-1
1	-2.72
2	-7.38

$\longrightarrow f(x) = -e^{x-3}$

x	f(x)
2	-.37
3	-1
4	-2.72
5	-7.38



D: $\mathbb{R}$

R: $\{y \mid y < 0\}$

Horizontal asymptote: $y = 0$

Solve exponential equations:

(use exponent rules plus following theorem)

If $a^u = a^v$ then $u = v$.

bases are identical.

<u>Example 7</u>: Solve $3^{x+1} = 81$ $\quad$ use above

$3^{x+1} = 3^4$ $\quad$ Know that $81 = 3^4$

$x+1 = 4$

$x = 3$ ✓

simplify before you solve.

Example 8: Solve $e^{-x^2} = (e^x)^2 \cdot \frac{1}{e^3}$

$$\boxed{x = -3 \qquad x = 1}$$

$$e^{-x^2} = e^{2x} \cdot e^{-3}$$

$$e^{-x^2} = e^{2x + -3}$$

$$-x^2 = 2x - 3$$

$$0 = x^2 + 2x - 3 \qquad \begin{array}{l} 3 \cdot -1 = -3 \\ 3 + -1 = 2 \end{array}$$

$$0 = (x+3)(x-1)$$

$$x + 3 = 0 \qquad x - 1 = 0$$

$$x = -3 \qquad x = 1$$

Example 9: Between 9:00 PM and 10:00 PM cars arrive at Burger King's drive-thru at the rate of 12 cars per hour (0.2 car minute). The following formula from statistics can be used to determine the probability that a car will arrive within t minutes of 9:00 PM.

$$F(t) = 1 - e^{-0.2t}$$

a) Determine the probability that a car will arrive within 5 minutes of 9 PM (that is before 9:05 PM).

$t = 5$

$$F(5) = 1 - e^{-.2(5)}$$

$$= 1 - e^{-1}$$

$$= 1 - \frac{1}{e} \approx .6321$$

There is a 63.21% that a car will arrive before 9:05 pm.

b) Determine the probability that a car will arrive within 30 minutes of 9 PM (that is before 9:30 PM).

$$F(30) = 1 - e^{-.2(30)}$$

$$= 1 - e^{-6}$$

$$= 1 - \frac{1}{e^6} \approx .9975$$

There is a 99.75% chance that a car will arrive before 9:30 pm.

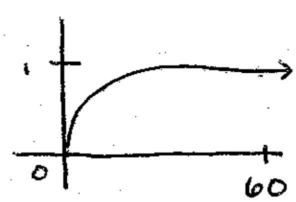
c) Graph F using your graphing utility.

d) What value does F approach as $\underbrace{t \text{ becomes unbounded in the positive direction}}_{t \to +\infty}$?

$$F \to ?$$

$y_{min} = 0$

$y_{max} = 1.25$



$x_{min} = 0$

$x_{max} = 60$

$$F(t) = 1 - e^{-.2t}$$

$$= 1 - \frac{1}{e^{.2t}} \longrightarrow \text{as } t \to +\infty \quad \left.\frac{1}{e^{.2t}}\right\} \to 0$$

$$= 1 - 0 = 1$$