5.2 One-to-One Functions; Inverse Functions

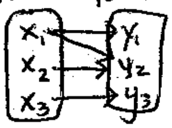
pairing b/t 2 sets

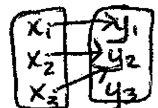
function – a relation where there is exactly 1 output for each input

① Table/List of ordered pairs ② Graph ③ Equation Rule

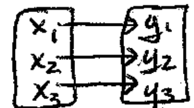
④ mapping diagram

relation NOT a function





Function



Function – 1 to 1

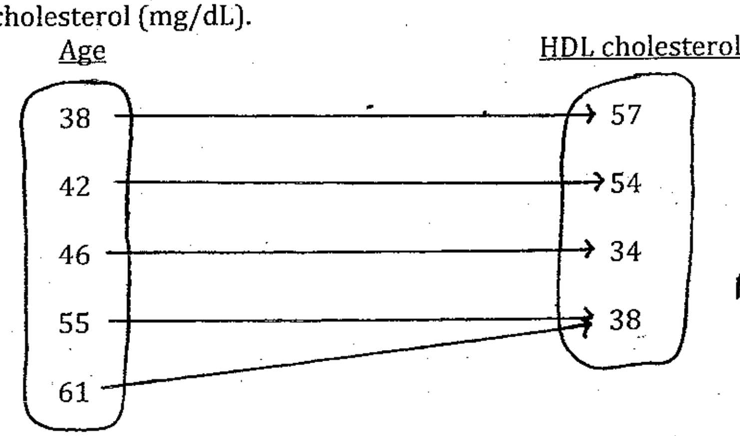
Definition of one–to-one:

A function where each output has exactly one input.

If x_1 and x_2 are different inputs of a function f, then f is one-to-one if $f(x_1) \neq f(x_2)$.

Example 1: Determine whether the following functions are one-to-one.

(a) For the following function, the domain represents the age of five males and the range represents their HDL (good) cholesterol (mg/dL).



No, because the output 38 has 2 inputs.

(b) $\{(-2,6), (-1,3,) (0,2), (1,5), (2,8)\}$

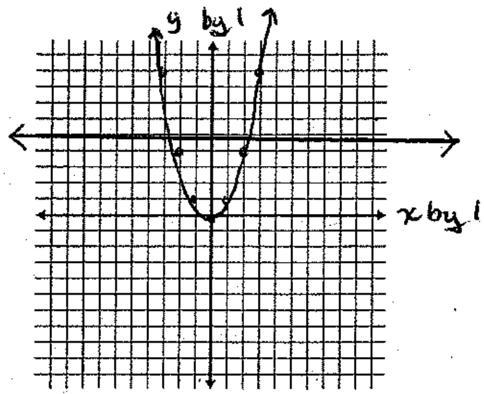
Yes, this function is one-to-one.

assume f a function

Horizontal Line Test – If every horizontal line intersects the graph of f in at most one point, then f is one-to-one.

Example 2: For each function, use its graph to determine whether the function is one-to-one.

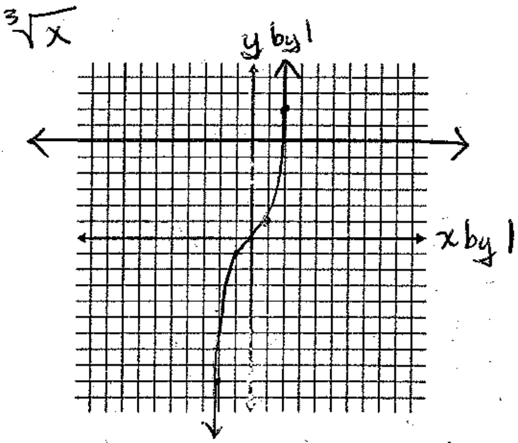
(a) $f(x) = x^2$



Not 1 to 1, fails HLT.

(b) $g(x) = x^3$

inverse $\sqrt[3]{x}$



This function is 1 to 1 b/c passes HLT.

Increasing/Decreasing and one-to-one:

A function that is increasing over an interval I is a 1-to-1 function on I.

" " " " decreasing " " " " " " " " ".

Definition of an inverse function:

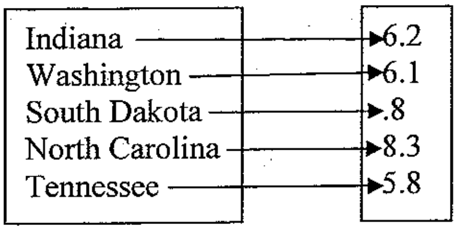
Suppose that a function f is one-to-one.

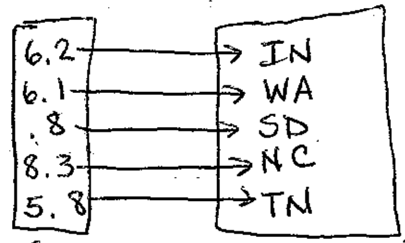
Then to each x in the domain of f, there is exactly 1 y in the range and to each y, there is exactly 1 x.

The correspondance (pairing/mapping) from the range BACK to the domain is called the inverse function of f.

We use $f^{-1}(x)$ to denote the inverse function.

Example 3: Find the inverse of the following function. Let the domain of the function represent certain states, and let the range represent the state's population (in millions). State the domain and the range of the inverse function.





D of f^{-1}: $\{6.2, 6.1, .8, 8.3, 5.8\}$

R of f^{-1}: {IN, WA, SD, NC, TN}

Example 4: Find the inverse of the following one-to-one function:

f $\{(-3,-27),(-2,-8),(-1,-1),(0,0),(1,1),(2,8),(3,27)\}$

f^{-1} $\{(-27,-3),(-8,-2),(-1,-1),(0,0),(1,1),(8,2),(27,3)\}$

Domain and Range of functions and their inverses:

D of $f \Rightarrow \{-3,-2,-1,0,1,2,3\} \Leftarrow$ R of f^{-1}

R of $f \Rightarrow \{-27,-8,-1,0,1,8,27\} \Leftarrow$ D of f^{-1}

To verify 2 functions are inverses, compose in both orders.

$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for all x in the D of $f^{-1}(x)$,

$(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for all x in the D of $f(x)$.

Example 5: (a) We verify that the inverse of $g(x) = x^3$ *is* $g^{-1}(x) = \sqrt[3]{x}$ by showing that

$g(g^{-1}(x)) = g(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$ for all x in the D of $g^{-1}(x)$.
$\mathbb{R}$

$g^{-1}(g(x)) = g^{-1}(x^3) = \sqrt[3]{x^3} = x$ for $\mathbb{R}$

(b) We verify that the inverse of $f(x) = 2x + 3$ *is* $f^{-1}(x) = \frac{1}{2}(x-3)$ by showing that

$f(f^{-1}(x)) = f(\frac{1}{2}(x-3)) = 2[\frac{1}{2}(x-3)] + 3 = x - 3 + 3 = x$ for all x in D of $f^{-1}(x)$.
for $\mathbb{R}$

$f^{-1}(f(x)) = f^{-1}(2x+3) = \frac{1}{2}[2x+3-3] = \frac{1}{2}[2x] = x$ for $\mathbb{R}$

Example 6: Verify that the inverse of $f(x) = \frac{1}{x-1}$ is $f^{-1}(x) = \frac{1}{x} + 1$. For what values of x is $f^{-1}(f(x)) = x$? For what values of x is $f(f^{-1}(x)) = x$?

$$f^{-1}(f(x)) = f^{-1}\left(\frac{1}{x-1}\right) = \frac{1}{\frac{1}{x-1}} + 1 = x-1+1 = x \quad \text{for } \{x \mid x \neq 1\}$$

$$f(f^{-1}(x)) = f\left(\frac{1}{x}+1\right) = \frac{1}{\frac{1}{x}+1-1} = \frac{1}{\frac{1}{x}} = x \quad \text{for } \{x \mid x \neq 0\}$$

The graph of an inverse function:

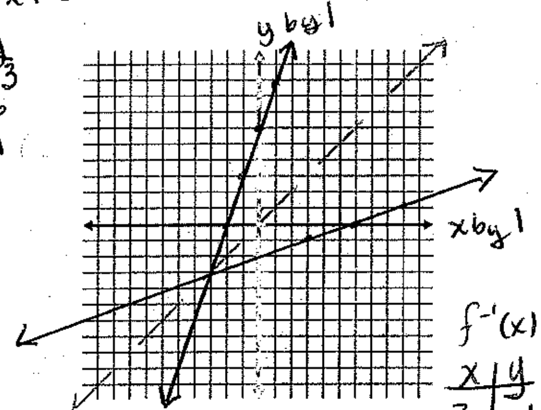
To graph an inverse...

have table? swap x and y

have the graph? reflect graph over the line $y=x$

$y = 3x+6$

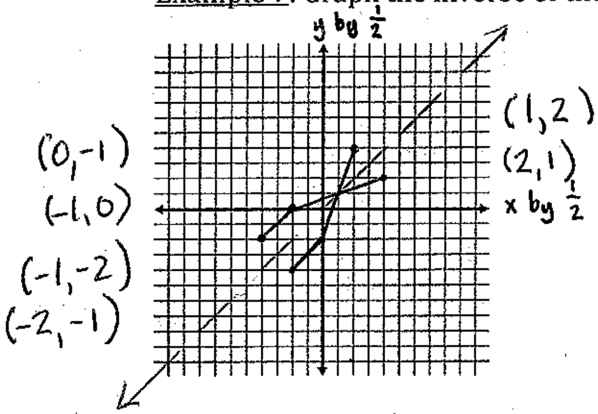
x	y
-1	3
0	6
1	9



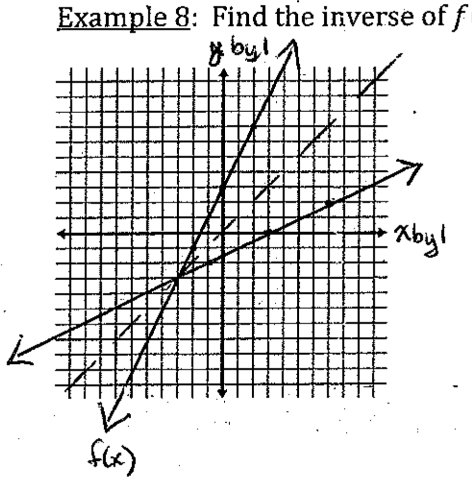
$f^{-1}(x)$

x	y
3	-1
6	0
9	1

Example 7: Graph the inverse of the one-to-one function graphed below.



Example 8: Find the inverse of $f(x) = 2x + 3$. Graph $f(x)$ and $f^{-1}(x)$ on the same coordinate axes.



$f(x) = 2x+3$

x	f(x)
-2	-1
0	3
2	7

$f^{-1}(x)$

x	f(x)
-1	-2
3	0
7	2

← the line?

$f^{-1}(x) = \frac{1}{2}x - \frac{3}{2}$

check algebraically.

$y = 2x+3$

$x = 2y+3$

$x-3 = 2y$

$\frac{1}{2}x - \frac{3}{2} = y$ ✓

(algebraically)

Procedure for Finding the Inverse of a One-to-One Function:

Step 1 : Replace $f(x)$ with y.
Swap the x and y. This is the implicit definition of the inverse.

Step 2 : If possible, solve for y.
Then replace the y with $f^{-1}(x)$. This is the explicit definition of the inverse.

Step 3 : Verify your result by composing the functions in both orders and get x (for all x in the domain)

* Example 9: The function

$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

v.a. $x = 1$ x int. $(-\frac{1}{2}, 0)$
h. a. $y = 2$ y-int. $(0, -1)$

is one-to-one. Find its inverse and check the result.

① $y = \frac{2x+1}{x-1}$

$\boxed{x = \frac{2y+1}{y-1} \Rightarrow \text{implicit def'n}}$

② $(y-1)x = \frac{2y+1}{\cancel{y-1}}\cancel{(y-1)}$

$(y-1)x = 2y+1$

$xy - x = 2y+1$

$xy - 2y - x = 1$

$xy - 2y = x+1$

$\frac{y(x-2)}{x-2} = \frac{x+1}{x-2} \rightarrow y = \frac{x+1}{x-2}$

h.a. $y=1$ y-int. $(0, -\frac{1}{2})$
v.a. $x=2$ x-int. $(-1, 0)$

$\boxed{\text{explicit def'n} \quad f^{-1}(x) = \frac{x+1}{x-2}}$

D: $\{x \mid x \neq 2\}$

③

$$f^{-1}(f(x)) = f^{-1}\left(\frac{2x+1}{x-1}\right) = \frac{\frac{2x+1}{x-1}+1}{\frac{2x+1}{x-1}-2} = \frac{\frac{2x+1}{x-1}+\frac{x-1}{x-1}}{\frac{2x+1}{x-1}-2\left(\frac{x-1}{x-1}\right)} = \frac{\frac{3x}{x-1}}{\frac{3}{x-1}} = \frac{\cancel{3}x}{\cancel{x-1}} \cdot \frac{\cancel{x-1}}{\cancel{3}} = x$$

for $\{x \mid x \neq 1\}$

$$f(f^{-1}(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{2\left(\frac{x+1}{x-2}\right)+1}{\left(\frac{x+1}{x-2}\right)-1} = \frac{\frac{2x+2+x-2}{x-2}}{\frac{x+1-(x-2)}{x-2}} = \frac{\frac{3x}{x-2}}{\frac{3}{x-2}} = \frac{\cancel{3}x}{\cancel{x-2}} \cdot \frac{\cancel{x-2}}{\cancel{3}} = x$$

for $\{x \mid x \neq 2\}$

Example 10: Find the domain and range of

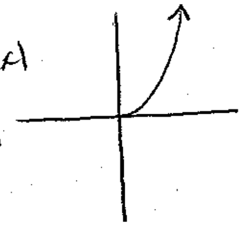
$$f(x) = \frac{2x+1}{x-1} \qquad x \neq 1$$

D: $\{x \mid x \neq 1\}$

R: $\{y \mid y \neq 2\}$

domain restricted so $f(x)$ is 1 to 1

<u>Example 11</u>: Find the inverse of $y = f(x) = x^2 \, if \, x \geq 0$.



$y = x^2$

$x = y^2$ implicit def'n

$\sqrt{x} = \sqrt{y^2}$

$\sqrt{x} = y$

$f^{-1}(x) = \sqrt{x}$ explicit def'n.

$f(f^{-1}(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$ for $\{x \mid x \geq 0\}$

$f^{-1}(f(x)) = f^{-1}(x^2) = \sqrt{x^2} = x$ for $\{x \mid x \geq 0\}$

Summary:

1. A function has an inverse iff it is one-to-one.
2. Domain of $f(x)$ = Range of $f^{-1}(x)$
   Range of $f(x)$ = Domain of $f^{-1}(x)$
3. To verify 2 functions are inverses, show that
   $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for all x in the domain of $f^{-1}(x)$.
   $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ " " " " " " of $f(x)$.
4. The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $y = x$.